LOWERING TOPOLOGICAL ENTROPY OVER SUBSETS

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Abstract. Let (X,T) be a topological dynamical system (TDS), and $h(T,K)$ the topological entropy of a subset K of X. (X,T) is *lowerable* if for each $0 \leq h \leq h(T,X)$ there is a non-empty compact subset with entropy h; is *hereditarily lowerable* if each non-empty compact subset is lowerable; is *hereditarily uniformly lowerable* if for each non-empty compact subset K and each $0 \leq h \leq h(T,K)$ there is a non-empty compact subset $K_h \subseteq K$ with $h(T,K_h)=h$ and K_h has at most one limit point.

It is shown that each TDS with finite entropy is lowerable, and that a TDS (X,T) is hereditarily uniformly lowerable if and only if it is asymptotically h-expansive.

1. Introduction

Throughout the paper, by a *topological dynamical system* (TDS) (X,T) we mean a compact metric space X and a homeomorphism $T: X \to X$ (in fact our main results hold for continuous maps, see Appendix). Let (X,T) be a TDS. It is an interesting question, considered in [28] firstly, whether for any given $0 \leq h \leq h_{\mathrm{top}}(T,X)$, there is a factor (Y,S) of (X,T) with entropy h. We remark that the answer to this question in the measure-theoretical setup is well known, but in the topological setting the answer is not completely obtained yet. In [28] Shub and Weiss presented an example with infinite entropy such that each its non-trivial factor has infinite entropy. Moreover, Lindenstrauss [20] showed that the question has an affirmative answer when X is finite-dimensional; and for an extension of non-trivial minimal $\mathbb{Z}$-actions the question has an affirmative answer if it has zero mean topological dimension [21] which includes finite-dimensional systems, systems with finite entropy and uniquely ergodic systems. For the definition and properties of mean topological dimension see [22] by Lindenstrauss and Weiss.

Let (X,T) be a TDS and $K \subseteq X$. Denote by $h(T,K)$ the topological entropy of K. In this paper we study a question similar to the above one. Namely, we consider the question if for each $0 \leq h \leq h(T,X)$ there is a non-empty compact subset of X with entropy h. We remark that the question was motivated by [28, 20, 22, 21] and the well-known result in fractal geometry [11, 23] which states that if K is a

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non-empty Borel subset contained in $\mathbb{R}^n$ then for each $0 \leq h \leq dim_H(K)$ there is a Borel subset K_h of K with $dim_H(K_h) = h$, where $dim_H(*)$ is the Hausdorff dimension of a subset $*$ of $\mathbb{R}^n$.

In [30] Ye and Zhang introduced and studied the notion of entropy points, and showed that for each non-empty compact subset K there is a countable compact subset $K_1 \subseteq K$ with $h(T, K_1) = h(T, K)$. Moreover, the subset can be chosen such that the limit points of the subset has at most one limit point (for details see [30, Remark 5.13]). Inspired by this fact we have the following notions.

Definition 1.1. Let (X, T) be a TDS. We say that (X, T) is

(1) *lowerable* if for each $0 \leq h \leq h(T, X)$ there is a non-empty compact subset of X with entropy h;
(2) *hereditarily lowerable* if each non-empty compact subset is lowerable, that is, for each non-empty compact subset $K \subseteq X$ and each $0 \leq h \leq h(T, K)$ there is a non-empty compact subset K_h of K with entropy h;
(3) *hereditarily uniformly lowerable* (**HUL** for short) if for each non-empty compact subset K and each $0 \leq h \leq h(T, K)$ there is a non-empty compact subset $K_h \subseteq K$ such that $h(T, K_h) = h$ and K_h has at most one limit point.

So our question can be divided further into the following questions.

Question 1.2. *Is any TDS lowerable?*

Question 1.3. *Is any TDS hereditarily lowerable?*

Question 1.4. *For which TDS it is* **HUL***?*

We remark that lowering entropy for factors is not the same as lowering entropy for subsets. For example, in [20] Lindenstrauss showed that each non-trivial factor of $([0, 1]^{\mathbb{Z}}, \sigma)$ has infinite entropy, where σ is the shift. But since $(\{0, 1, \ldots, k\}^{\mathbb{Z}}, \sigma)$ can be embedded as a sub-system of $([0, 1]^{\mathbb{Z}}, \sigma)$ for any $k \geq 1$, it is clear that $([0, 1]^{\mathbb{Z}}, \sigma)$ is lowerable in our sense.

In this paper, we show that each TDS with finite entropy is lowerable (this is also true when we talk about the dimensional entropy of a subset), and that a TDS is **HUL** iff it is asymptotically h-expansive. In particular, each **HUL** TDS has finite entropy. Moreover, a principal extension preserves the lowerable, hereditarily lowerable and **HUL** properties. It is not hard to construct examples with infinite entropy which are hereditarily lowerable. Thus, there are TDSs which are hereditarily lowerable but not **HUL**. In fact, an example with the same property is explored at the end of the paper, which has finite entropy. The questions remain open if there are lowerable but not hereditarily lowerable examples, or there are TDSs with infinite entropy which are not lowerable. We should remark that if $([0, 1]^{\mathbb{Z}}, \sigma)$ is hereditarily lowerable then each finite dimensional TDS without periodic points is hereditarily lowerable (see [21]), and if it is not then it is a lowerable TDS with infinite entropy which is not hereditarily lowerable. We also remark that if there exists a TDS which is not lowerable (such a TDS, if exists, must have infinite entropy) then we can obtain a lowerable TDS with infinite entropy which is not hereditarily lowerable by

considering the union of it and $([0,1]^{\mathbb{Z}}, \sigma)$. There are also many other interesting questions related to the topic.

The paper is organized as follows. In section 2 the definitions of topological entropy and dimensional entropy of subsets are given, and some basic properties are discussed. In the following section two distribution principles are stated which will be used in section 4, where it is shown that each TDS with finite entropy is lowerable by using the principles and a conditional version of Shannon-McMillan-Breiman Theorem. The next three sections are devoted to prove that a TDS is **HUL** iff it is asymptotically h-expansive, and the main ingredients of which are some techniques developed in [30, 5, 10, 19]. An example with finite entropy which is hereditarily lowerable but not **HUL** is presented at the end of paper.

We thank D. Feng [12] for asking the question: whether each non-empty compact subset is lowerable? His question gave us the first motivation of the research. We also thank the referees of the paper for their careful reading and useful suggestions which greatly improved the writing of the paper.

2. Preliminary

The discussions in this section and next section proceed for a *general TDS* (GTDS), by a GTDS (X,T) we mean a compact metric space X and a continuous mapping $T : X \to X$.

Let (X,T) be a GTDS, $K \subseteq X$ and $\mathcal{W}$ a family of subsets of X. Set $\mathrm{diam}(K)$ to be the diameter of K and put $||\mathcal{W}|| = \sup\{\mathrm{diam}(W) : W \in \mathcal{W}\}$. We shall write $K \succeq \mathcal{W}$ if $K \subseteq W$ for some $W \in \mathcal{W}$ and else $K \not\succeq \mathcal{W}$. If $\mathcal{W}_1$ is another family of subsets of X, $\mathcal{W}$ is said to be *finer* than $\mathcal{W}_1$ (we shall write $\mathcal{W} \succeq \mathcal{W}_1$) when $W \succeq \mathcal{W}_1$ for each $W \in \mathcal{W}$. We shall say that a numerical function *increases* (resp. *decreases*) with respect to (w.r.t.) a set variable K or a family variable $\mathcal{W}$ if the value never decreases (resp. increases) when K is replaced by a set K_1 with $K_1 \subseteq K$ or when $\mathcal{W}$ is replaced by a family $\mathcal{W}_1$ with $\mathcal{W}_1 \succeq \mathcal{W}$. By a *cover* of X we mean a finite family of Borel subsets with union X and a *partition* a cover whose elements are disjoint. Denote by $\mathcal{C}_X$ (resp. $\mathcal{C}_X^o$, $\mathcal{P}_X$) the set of covers (resp. open covers, partitions). Observe that if $\mathcal{U} \in \mathcal{C}_X^o$ then $\mathcal{U}$ has a Lebesgue number $\lambda > 0$ and so $\mathcal{W} \succeq \mathcal{U}$ when $||\mathcal{W}|| < \lambda$. If $\alpha \in \mathcal{P}_X$ and $x \in X$ then let $\alpha(x)$ be the element of α containing x. Given $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{C}_X$, set $\mathcal{U}_1 \vee \mathcal{U}_2 = \{U_1 \cap U_2 : U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2\}$, obviously $\mathcal{U}_1 \vee \mathcal{U}_2 \in \mathcal{C}_X$ and $\mathcal{U}_1 \vee \mathcal{U}_2 \succeq \mathcal{U}_1$. $\mathcal{U}_1 \succeq \mathcal{U}_2$ need not imply that $\mathcal{U}_1 \vee \mathcal{U}_2 = \mathcal{U}_1$, $\mathcal{U}_1 \succeq \mathcal{U}_2$ iff $\mathcal{U}_1$ is equivalent to $\mathcal{U}_1 \vee \mathcal{U}_2$ in the sense that each refines the other. For each $\mathcal{U} \in \mathcal{C}_X$ and any $m, n \in \mathbb{Z}_+$ with $m \leq n$ we set $\mathcal{U}_m^n = \bigvee_{i=m}^n T^{-i}\mathcal{U}$.

The following obvious fact will be used in several places and is easy to check.

Lemma 2.1. *Let* $\mathcal{V} \in \mathcal{C}_X^o$ *and* $\{\mathcal{U}_n : n \in \mathbb{N}\} \subseteq \mathcal{C}_X$. *If* $||\mathcal{U}_n|| \to 0$ *as* $n \to +\infty$ *then there exists* $n_0 \in \mathbb{N}$ *such that* $\mathcal{U}_n \succeq \mathcal{V}$ *for each* $n \geq n_0$.

2.1. **Topological entropy of subsets.**

Let (X,T) be a GTDS, $K \subseteq X$ and $\mathcal{U} \in \mathcal{C}_X$. Set $N(\mathcal{U}, K)$ to be the minimal cardinality of sub-families $\mathcal{V} \subseteq \mathcal{U}$ with $\cup\mathcal{V} \supseteq K$, where $\cup\mathcal{V} = \bigcup_{V\in\mathcal{V}} V$. We write $N(\mathcal{U}, \emptyset) = 0$ by convention. Obviously, $N(\mathcal{U}, T(K)) = N(T^{-1}\mathcal{U}, K)$. Let

$$h_{\mathcal{U}}(T,K) = \limsup_{n\to+\infty} \frac{1}{n} \log N(\mathcal{U}_0^{n-1}, K).$$

Clearly $h_{\mathcal{U}}(T,K)$ increases w.r.t. $\mathcal{U}$. Define the *topological entropy of* K by

$$h(T,K) = \sup_{\mathcal{U}\in\mathcal{C}_X^o} h_{\mathcal{U}}(T,K),$$

and define the *topological entropy of* (X,T) by $h_{\text{top}}(T,X) = h(T,X)$.

Let Z be a topological metric space and $f : Z \to [-\infty, +\infty]$ a generalized real-valued function on Z. The function f is called *upper semi-continuous* (u.s.c. for short) if $\{z \in Z : f(z) \geq r\}$ is a closed subset of Z for each $r \in \mathbb{R}$, equivalently,

$$\limsup_{z'\to z} f(z') \leq f(z) \text{ for each } z \in Z.$$

Thus, the infimum of any family of u.s.c. functions is again a u.s.c. one, both the sum and supremum of finitely many u.s.c. functions are u.s.c. ones. In particular, the infimum of any family of continuous functions is a u.s.c. function.

Let (X,T) be a GTDS and 2^X its hyperspace, that is,

$$2^X = \{K : K \text{ is a non-empty compact subset of } X\}.$$

We endow the Hausdorff metric on 2^X. Then T induces a continuous mapping $\widehat{T}$ on 2^X by $\widehat{T}(K) = TK$. The *entropy hyper-function* $H : 2^X \to [0, h_{\text{top}}(T,X)]$ of (X,T) is defined by $H(K) = h(T,K)$ for $K \in 2^X$. Then we have the follow results.

Proposition 2.2. *Let* (X,T) *be a GTDS and* $\mathcal{U} \in \mathcal{C}_X^o$. *Then*

1. $h_{\mathcal{U}}(T,K) = h_{\mathcal{U}}(T,TK)$ *for any* $K \subseteq X$. *Moreover, the entropy hyper-function* H *is* $\widehat{T}$*-invariant.*
2. *The function* $h_{\mathcal{U}}(T,\bullet)$ *is Borel measurable on* 2^X.
3. *The entropy hyper-function* H *is Borel measurable.*

Proof. (1) is clear. (2) follows from the following fact that for any $\mathcal{V} \in \mathcal{C}_X^o$, $N(\mathcal{V}, \bullet) : K \in 2^X \mapsto N(\mathcal{V}, K)$ is a u.s.c function on 2^X. (3) comes from (2). $\square$

We may also obtain the topological entropy of subsets using Bowen's separated and spanning sets (see [29, $\text{P}_{168-174}$]). Let (X,T) be a TDS with d a metric on X. For each $n \in \mathbb{N}$ we define a new metric d_n on X by

$$d_n(x,y) = \max_{0\leq i\leq n-1} d(T^i x, T^i y).$$

Let $\epsilon > 0$ and $K \subseteq X$. A subset F of X is said to (n,ϵ)*-span* K *w.r.t.* T if for each $x \in K$, there is $y \in F$ with $d_n(x,y) \leq \epsilon$; a subset E of K is said to be (n,ϵ)*-separated w.r.t.* T if $x, y \in E, x \neq y$ implies $d_n(x,y) > \epsilon$. Let $r_n(d,T,\epsilon,K)$ denote the smallest cardinality of any (n,ϵ)-spanning set for K w.r.t. T and $s_n(d,T,\epsilon,K)$

denote the largest cardinality of any (n, ϵ)-separated subset of K w.r.t. T. We write $r_n(d, T, \epsilon, \emptyset) = s_n(d, T, \epsilon, \emptyset) = 0$ by convention. Put

$$r(d, T, \epsilon, K) = \limsup_{n \to +\infty} \frac{1}{n} \log r_n(d, T, \epsilon, K)$$

and

$$s(d, T, \epsilon, K) = \limsup_{n \to +\infty} \frac{1}{n} \log s_n(d, T, \epsilon, K).$$

Then put

$$h_*(d, T, K) = \lim_{\epsilon \to 0+} r(d, T, \epsilon, K) \text{ and } h^*(d, T, K) = \lim_{\epsilon \to 0+} s(d, T, \epsilon, K).$$

It is well known that $h_*(d, T, K) = h^*(d, T, K)$ is independent of the choice of a compatible metric d on the space X. Now, if $\mathcal{U} \in \mathcal{C}_X^o$ has a Lebesgue number $\delta > 0$ then, for any $\delta' \in (0, \frac{\delta}{2})$ and each $\mathcal{V} \in \mathcal{C}_X^o$ with $||\mathcal{V}|| \le \delta'$, one has

$$N(\mathcal{U}_0^{n-1}, K) \le r_n(d, T, \delta', K) \le s_n(d, T, \delta', K) \le N(\mathcal{V}_0^{n-1}, K)$$

for each $n \in \mathbb{N}$. So if $\{\mathcal{U}_n\}_{n \in \mathbb{N}} \subseteq \mathcal{C}_X^o$ satisfies $||\mathcal{U}_n|| \to 0$ as $n \to +\infty$ then

$$h_*(d, T, K) = h^*(d, T, K) = \lim_{n \to +\infty} h_{\mathcal{U}_n}(T, K) = h(T, K).$$

It is also obvious that $h(T, \overline{K}) = h(T, K)$.

2.2. **Dimensional entropy of subsets.**

In the process of proving that each TDS with finite entropy is lowerable, we shall use some concept named dimensional entropy of subsets, which is another kind of topological entropy introduced and studied in [4]. Let's see how to define it.

Let (X, T) be a GTDS and $\mathcal{U} \in \mathcal{C}_X$. For $K \subseteq X$ let

$$n_{T,\mathcal{U}}(K) = \begin{cases} 0, & \text{if } K \not\succeq \mathcal{U}; \\ +\infty, & \text{if } T^i K \succeq \mathcal{U} \text{ for all } i \in \mathbb{Z}_+; \\ k, & k = \max\{j \in \mathbb{N} : T^i(K) \succeq \mathcal{U} \text{ for each } 0 \le i \le j-1\}. \end{cases}$$

For $k \in \mathbb{N}$, we define

$$\begin{aligned} \mathfrak{C}(T, \mathcal{U}, K, k) \;\; = \;\; & \{\mathcal{E} : \mathcal{E} \text{ is a countable family of subsets of } X \\ & \qquad \text{such that} K \subseteq \cup \mathcal{E} \text{ and } \mathcal{E} \succeq \mathcal{U}_0^{k-1}\}. \end{aligned}$$

Then for each $\lambda \in \mathbb{R}$ set

$$m_{T,\mathcal{U}}(K, \lambda, k) = \inf_{\mathcal{E} \in \mathfrak{C}(T, \mathcal{U}, K, k)} m(T, \mathcal{U}, \mathcal{E}, \lambda),$$

where $m(T, \mathcal{U}, \mathcal{E}, \lambda) = \sum_{E \in \mathcal{E}} e^{-\lambda n_{T,\mathcal{U}}(E)}$ and we write $m(T, \mathcal{U}, \emptyset, \lambda) = 0$ by convention. As $m_{T,\mathcal{U}}(K, \lambda, k)$ is decreasing w.r.t. k, we can define

$$m_{T,\mathcal{U}}(K, \lambda) = \lim_{k \to +\infty} m_{T,\mathcal{U}}(K, \lambda, k).$$

Notice that $m_{T,\mathcal{U}}(K, \lambda) \le m_{T,\mathcal{U}}(K, \lambda')$ for $\lambda \ge \lambda'$ and $m_{T,\mathcal{U}}(K, \lambda) \notin \{0, +\infty\}$ for at most one λ [4]. We define *the dimensional entropy of K relative to $\mathcal{U}$* by

$$h_{\mathcal{U}}^B(T, K) = \inf\{\lambda \in \mathbb{R} : m_{T,\mathcal{U}}(K, \lambda) = 0\} = \sup\{\lambda \in \mathbb{R} : m_{T,\mathcal{U}}(K, \lambda) = +\infty\}.$$

The *dimensional entropy of* K is defined by
$$h^B(T,K)=\sup_{\mathcal{U}\in\mathcal{C}_X^o} h^B_{\mathcal{U}}(T,K).$$
Note that $h^B_{\mathcal{U}}(T,K)$ increases w.r.t. $\mathcal{U}\in\mathcal{C}_X$, thus if $\{\mathcal{U}_n\}_{n\in\mathbb{N}}\subseteq\mathcal{C}_X^o$ satisfies $\lim_{n\to+\infty}||\mathcal{U}_n||=0$ then $\lim_{n\to+\infty}h^B_{\mathcal{U}_n}(T,K)=h^B(T,K)$.

The following results are elementary (see for example [4, Propositions 1 and 2]).

Proposition 2.3. *Let* (X,T) *be a GTDS,* $K_1,K_2,\cdots,K\subseteq X$ *and* $\mathcal{U}\in\mathcal{C}_X$. *Then*

(1) $h_{\mathcal{U}}(T,X)=h^B_{\mathcal{U}}(T,X)$ *if* $\mathcal{U}\in\mathcal{C}_X^o$, *so* $h(T,X)=h^B(T,X)$.
(2) $h^B_{\mathcal{U}}(T,\bigcup_{n\in\mathbb{N}}K_n)=\sup_{n\in\mathbb{N}}h^B_{\mathcal{U}}(T,K_n)$, *so*
$$h^B(T,\bigcup_{n\in\mathbb{N}}K_n)=\sup_{n\in\mathbb{N}}h^B(T,K_n).$$
(3) *For each* $m\in\mathbb{N}$ *and* $i\geq 0$, $h^B_{T^{-i}\mathcal{U}}(T^m,K)\geq h^B_{\mathcal{U}}(T^m,T^iK)$, *so* $h^B(T^m,K)\geq h^B(T^m,T^iK)$.
(4) *For each* $m\in\mathbb{N}$, $h^B_{\mathcal{U}_0^{m-1}}(T^m,K)=mh^B_{\mathcal{U}}(T,K)$, *so* $h^B(T^m,K)=mh^B(T,K)$.

Proof. (1) is [4, Proposition 1]. (2) is obvious.

(3) Let $m\in\mathbb{N}$ and $i\geq 0$. Assume $k\in\mathbb{N}$ and $\lambda>0$. If $\mathcal{E}\in\mathfrak{C}(T^m,T^{-i}\mathcal{U},K,k)$ then $n_{T^m,\mathcal{U}}(T^iE)=n_{T^m,T^{-i}\mathcal{U}}(E)\geq k$ for each $E\in\mathcal{E}$ and so
$$T^i(\mathcal{E})\doteq\{T^iE:E\in\mathcal{E}\}\in\mathfrak{C}(T^m,\mathcal{U},T^iK,k),$$
thus
$$\begin{aligned} m_{T^m,\mathcal{U}}(T^iK,\lambda,k) &\leq m(T^m,\mathcal{U},T^i(\mathcal{E}),\lambda)=\sum_{E\in\mathcal{E}}e^{-\lambda n_{T^m,\mathcal{U}}(T^iE)}\\ &= \sum_{E\in\mathcal{E}}e^{-\lambda n_{T^m,T^{-i}\mathcal{U}}(E)}=m(T^m,T^{-i}\mathcal{U},\mathcal{E},\lambda),\end{aligned}$$
which implies $m_{T^m,\mathcal{U}}(T^iK,\lambda,k)\leq m_{T^m,T^{-i}\mathcal{U}}(K,\lambda,k)$ as $\mathcal{E}$ is arbitrary. Letting $k\to+\infty$ we get $m_{T^m,\mathcal{U}}(T^iK,\lambda)\leq m_{T^m,T^{-i}\mathcal{U}}(K,\lambda)$, hence $h^B_{\mathcal{U}}(T^m,T^iK)\leq h^B_{T^{-i}\mathcal{U}}(T^m,K)$, as $\lambda>0$ is arbitrary.

(4) Let $m\in\mathbb{N}$ and $n\in\mathbb{N}$, $\lambda>0$. If $\mathcal{E}\in\mathfrak{C}(T,\mathcal{U},K,mn)$ then
$$n_{T^m,\mathcal{U}_0^{m-1}}(E)=\left[\frac{n_{T,\mathcal{U}}(E)}{m}\right]\geq\max\left\{n,\frac{n_{T,\mathcal{U}}(E)}{m}-\frac{m-1}{m}\right\}$$
for each $E\in\mathcal{E}$, where $[a]$ denotes the integral part of a real number a, so
$$\inf_{E\in\mathcal{E}}n_{T^m,\mathcal{U}_0^{m-1}}(E)\geq n,$$
thus $\mathcal{E}\in\mathfrak{C}(T^m,\mathcal{U}_0^{m-1},K,n)$ and
$$\begin{aligned} m_{T^m,\mathcal{U}_0^{m-1}}(K,\lambda,n) &\leq m(T^m,\mathcal{U}_0^{m-1},\mathcal{E},\lambda)=\sum_{E\in\mathcal{E}}(e^\lambda)^{-n_{T^m,\mathcal{U}_0^{m-1}}(E)}\\ &\leq \sum_{E\in\mathcal{E}}(e^\lambda)^{\frac{m-1}{m}-\frac{n_{T,\mathcal{U}}(E)}{m}}=e^{\frac{(m-1)\lambda}{m}}\cdot m(T,\mathcal{U},\mathcal{E},\frac{\lambda}{m}),\end{aligned}$$

which implies $m_{T^m,\mathcal{U}_0^{m-1}}(K,\lambda,n) \le e^{\frac{(m-1)\lambda}{m}} m_{T,\mathcal{U}}(K,\frac{\lambda}{m},mn)$ as $\mathcal{E}$ is arbitrary. We get

$$m_{T^m,\mathcal{U}_0^{m-1}}(K,\lambda) \le e^{\frac{(m-1)\lambda}{m}} \cdot m_{T,\mathcal{U}}(K,\frac{\lambda}{m})$$

by letting $n \to +\infty$, hence $h^B_{\mathcal{U}_0^{m-1}}(T^m,K) \le m h^B_{\mathcal{U}}(T,K)$, as $\lambda > 0$ is arbitrary.

Following similar discussions we obtain $m_{T,\mathcal{U}}(K,\lambda) \le m_{T^m,\mathcal{U}_0^{m-1}}(K,m\lambda)$ for each $\lambda > 0$, then $h^B_{\mathcal{U}}(T,K) \le \frac{1}{m} h^B_{\mathcal{U}_0^{m-1}}(T^m,K)$. That is, $h^B_{\mathcal{U}_0^{m-1}}(T^m,K) = m h^B_{\mathcal{U}}(T,K)$. □

By Proposition 2.3 (2), $h^B(T,E)$ increases w.r.t. $E \subseteq X$. At the same time, if $E \subseteq X$ is a non-empty countable set then $h^B(T,E) = 0$. Finally, it is worth mentioning that a). $h^B_{\mathcal{U}}(T,\emptyset) = h_{\mathcal{U}}(T,\emptyset) = -\infty$ for any $\mathcal{U} \in \mathcal{C}_X$, and so $h^B(T,\emptyset) = h(T,\emptyset) = -\infty$; b). when $\emptyset \ne K \subseteq X$, one has $h_{\mathcal{U}}(T,K) \ge h^B_{\mathcal{U}}(T,K) \ge 0$ for any $\mathcal{U} \in \mathcal{C}_X$, and so $h(T,K) \ge h^B(T,K) \ge 0$.

3. Distribution principles

In this section we shall present two important distribution principles which link Question 1.2 with ergodic theory and play a key role in the next section. We remark that the distribution principles were essentially contained in [26].

The first result is an obvious link between two definitions of entropy.

Lemma 3.1 (**Bridge Lemma**). *Let (X,T) be a GTDS, $\mathcal{U} \in \mathcal{C}_X$ and $K \subseteq X$. Then*

$$h^B_{\mathcal{U}}(T,K) \le \liminf_{n\to+\infty} \frac{1}{n} \log N(\mathcal{U}_0^{n-1},K) \le h_{\mathcal{U}}(T,K).$$

Proof. When $K = \emptyset$, this is clear. Now we assume $K \ne \emptyset$. For each $n \in \mathbb{N}$ let $\mathcal{T}_n = \{A_1,\cdots,A_{N(\mathcal{U}_0^{n-1},K)}\} \subseteq \mathcal{U}_0^{n-1}$ such that $\cup\mathcal{T}_n \supseteq K$. As $n_{T,\mathcal{U}}(A) \ge n$ for each $A \in \mathcal{T}_n$, for each $\lambda \ge 0$ one has

$$m_{T,\mathcal{U}}(K,\lambda,n) \le \sum_{A\in\mathcal{T}_n} \left(e^{\lambda}\right)^{-n_{T,\mathcal{U}}(A)} \le \sum_{A\in\mathcal{T}_n} (e^{\lambda})^{-n} = N(\mathcal{U}_0^{n-1},K)e^{-\lambda n},$$

then

$$m_{T,\mathcal{U}}(K,\lambda) \le \liminf_{n\to+\infty} N(\mathcal{U}_0^{n-1},K)e^{-\lambda n} = \liminf_{n\to+\infty} e^{-n(\lambda-\frac{1}{n}\log N(\mathcal{U}_0^{n-1},K))}.$$

So, if $\lambda > \liminf_{n\to+\infty} \frac{1}{n}\log N(\mathcal{U}_0^{n-1},K)$ then $m_{T,\mathcal{U}}(K,\lambda) = 0$, which ends the proof. □

Let (X,T) be a GTDS, $\mathcal{U} \in \mathcal{C}_X$, $K \subseteq X$ and $n \in \mathbb{N}$. Set $\mathfrak{M}(T,\mathcal{U},K,n)$ to be the collection of all countable families $\mathcal{T}$ of subsets of X with

$$\cup\mathcal{T} \supseteq K \text{ and for each } A \in \mathcal{T}, A\cap K \ne \emptyset, n_{T,\mathcal{U}}(A) \ge n \text{ and } A \in \mathcal{U}_0^{n_{T,\mathcal{U}}(A)-1}.$$

Then for each $\lambda \in \mathbb{R}$ set

$$f_{T,\mathcal{U}}(K,\lambda) = \lim_{n\to+\infty} \inf_{\mathcal{T}\in\mathfrak{M}(T,\mathcal{U},K,n)} m(T,\mathcal{U},\mathcal{T},\lambda).$$

It's not hard to check that $f_{T,\mathcal{U}}(K,\lambda) = m_{T,\mathcal{U}}(K,\lambda)$ for $\lambda \in \mathbb{R}$. In fact, for $\mathcal{E} \in \mathfrak{C}(T,\mathcal{U},K,n)$, note that for each $E \in \mathcal{E}$ there exists $\widetilde{E} \in \mathcal{U}_0^{n_{T,\mathcal{U}}(E)-1}$ with $E \subseteq$

$\widetilde{E}$ and so $n_{T,\mathcal{U}}(E) = n_{T,\mathcal{U}}(\widetilde{E})$. Then let $\mathcal{T} \doteq \{\widetilde{E} : E \in \mathcal{E} \text{ with } E \cap K \neq \emptyset\}$. Then $\mathcal{T} \in \mathfrak{M}(T, \mathcal{U}, K, n)$. Particularly, when $E \cap K \neq \emptyset$ for each $E \in \mathcal{E}$, one has $m(T, \mathcal{U}, \mathcal{T}, \lambda) = m(T, \mathcal{U}, \mathcal{E}, \lambda)$. This implies $f_{T,\mathcal{U}}(K, \lambda) = m_{T,\mathcal{U}}(K, \lambda)$.

For a GTDS (X, T), denote by $\mathcal{M}(X)$ the set of all Borel probability measures on X. The following two principles will be proved to be very useful.

Lemma 3.2 (**Non-Uniform Mass Distribution Principle).** *Let (X, T) be a GTDS, $d > 0, M \in \mathbb{N}$, $Z \subseteq X$, $\alpha \in \mathcal{P}_X, \mathcal{U} \in \mathcal{C}_X$ and $\theta \in \mathcal{M}(X)$. Assume that each element of $\mathcal{U}$ has a non-empty intersection with at most M elements of α. If there exists $Z_\theta \subseteq Z$ such that Z_θ has positive outer θ-measure (i.e. $\theta^*(Z_\theta) > 0$) and*

$$\forall x \in Z_\theta, \exists c(x) > 0 \text{ such that } \forall n \in \mathbb{N}, \theta(\alpha_0^{n-1}(x)) \leq c(x) e^{-nd}.$$

Then $h_{\mathcal{U}}^B(T, Z) \geq d - \log M$. In particular, $h_\alpha^B(T, Z) \geq d$.

Proof. It makes no difference to assume $d - \log M > 0$. For each $k \in \mathbb{N}$ set $Z_\theta^k = \{x \in Z_\theta : c(x) \leq k\}$. Then for some $N \in \mathbb{N}$, $\theta^*(Z_\theta^N) > 0$, as $Z_\theta^1 \subseteq Z_\theta^2 \subseteq \cdots$, $Z_\theta = \bigcup_{k \in \mathbb{N}} Z_\theta^k$ and $\theta^*(Z_\theta) > 0$.

Let $n \in \mathbb{N}$ and $\mathcal{T} \in \mathfrak{M}(T, \mathcal{U}, Z, n)$. If $A \in \mathcal{T}$ satisfies $A \cap Z_\theta^N \neq \emptyset$, then for each $s \in \mathbb{N}$ and $B \in \alpha_0^{\min\{n_{T,\mathcal{U}}(A), s\}-1}$ with $B \cap (A \cap Z_\theta^N) \neq \emptyset$ select $x_B \in B \cap (A \cap Z_\theta^N)$, so

$$\theta(B) = \theta(\alpha_0^{\min\{n_{T,\mathcal{U}}(A), s\}-1}(x_B)) \leq c(x_B) e^{-\min\{n_{T,\mathcal{U}}(A), s\}d} \leq N e^{-\min\{n_{T,\mathcal{U}}(A), s\}d}.$$

Since there are at most $M^{\min\{n_{T,\mathcal{U}}(A), s\}}$ elements of $\alpha_0^{\min\{n_{T,\mathcal{U}}(A), s\}-1}$ which have non-empty intersection with $A \cap Z_\theta^N$, we have

$$\theta^*(A \cap Z_\theta^N) \leq M^{\min\{n_{T,\mathcal{U}}(A), s\}} N e^{-\min\{n_{T,\mathcal{U}}(A), s\}d} = N e^{-\min\{n_{T,\mathcal{U}}(A), s\}(d - \log M)}.$$

Letting $s \to +\infty$ we obtain that $\theta^*(A \cap Z_\theta^N) \leq N e^{-n_{T,\mathcal{U}}(A)(d - \log M)}$ for any $A \in \mathcal{T}$ satisfying $A \cap Z_\theta^N \neq \emptyset$. Moreover,

$$\begin{aligned} \sum_{A \in \mathcal{T}} e^{-n_{T,\mathcal{U}}(A)(d - \log M)} &\geq \sum_{A \in \mathcal{T}, A \cap Z_\theta^N \neq \emptyset} e^{-n_{T,\mathcal{U}}(A)(d - \log M)} \\ &\geq \sum_{A \in \mathcal{T}, A \cap Z_\theta^N \neq \emptyset} \frac{\theta^*(A \cap Z_\theta^N)}{N} \geq \frac{1}{N} \theta^*(Z_\theta^N) > 0. \end{aligned}$$

Since n and $\mathcal{T}$ are arbitrary, $f_{T,\mathcal{U}}(Z, d - \log M) \geq \frac{1}{N} \theta^*(Z_\theta^N) > 0$. So $h_{\mathcal{U}}^B(T, Z) \geq d - \log M$. $\square$

Lemma 3.3 (**Uniform Mass Distribution Principle).** *Let (X, T) be a GTDS, $c > 0$, $d > 0$, $Z \subseteq X$ and $\alpha \in \mathcal{P}_X$. If there exists $\theta \in \mathcal{M}(X)$ such that for each $x \in Z$ and $n \in \mathbb{N}$, $\theta(\alpha_0^{n-1}(x)) \geq c e^{-nd}$. Then $h_\alpha(T, Z) \leq d$.*

Proof. If $Z = \emptyset$ then $h_\alpha(T, Z) = -\infty \leq d$. In the following we assume $Z \neq \emptyset$. For $n \in \mathbb{N}$, let $\mathcal{T}_n = \{A_1, \cdots, A_k\}$ be the collection of all elements of α_0^{n-1} which have non-empty intersection with Z, where $k = N(\alpha_0^{n-1}, Z)$. Take $x_i \in A_i \cap Z$, then $\theta(A_i) = \theta(\alpha_0^{n-1}(x_i)) \geq c e^{-nd}$ for $i \in \{1, \cdots, k\}$. Therefore $1 \geq \sum_{i=1}^k \theta(A_i) \geq k c e^{-nd}$, that is, $N(\alpha_0^{n-1}, Z) = k \leq \frac{e^{nd}}{c}$. Finally letting $n \to +\infty$ we know $h_\alpha(T, Z) \leq d$. $\square$

4. Each TDS with finite entropy is lowerable

In this section we shall give an affirmative answer to Question 1.2 for a TDS with finite entropy. In fact, we can obtain more about it. Precisely, if (X, T) is a TDS with finite entropy, then for each $0 \leq h \leq h_{\text{top}}(T, X)$ there exists a non-empty compact subset $K_h \subseteq X$ such that $h^B(T, K_h) = h(T, K_h) = h$ (for details see Theorem 4.4), particularly, (X, T) is lowerable.

Let (X, T) be a GTDS. Denote by $\mathcal{M}(X, T)$ and $\mathcal{M}^e(X, T)$ the set of all T-invariant Borel probability measures and ergodic T-invariant Borel probability measures on X, respectively. Then $\mathcal{M}(X)$ and $\mathcal{M}(X, T)$ are both convex, compact metric spaces when endowed with the weak*-topology. Denote by $\mathcal{B}_X$ the set of all Borel subsets of X.

For any given $\alpha \in \mathcal{P}_X$, $\mu \in \mathcal{M}(X)$ and any sub-σ-algebra $\mathcal{C} \subseteq \mathcal{B}_\mu$, where $\mathcal{B}_\mu$ is the completion of $\mathcal{B}_X$ under μ, the *conditional informational function of α relevant to $\mathcal{C}$* is defined by

$$I_\mu(\alpha|\mathcal{C})(x) = \sum_{A \in \alpha} -1_A(x) \log \mathbb{E}_\mu(1_A|\mathcal{C})(x),$$

where $\mathbb{E}_\mu(1_A|\mathcal{C})$ is the conditional expectation of 1_A w.r.t. $\mathcal{C}$. Let

$$H_\mu(\alpha|\mathcal{C}) = \int_X I_\mu(\alpha|\mathcal{C})(x)\, d\mu(x) = \sum_{A \in \alpha} \int_X -\mathbb{E}_\mu(1_A|\mathcal{C}) \log \mathbb{E}_\mu(1_A|\mathcal{C}) d\mu.$$

A standard fact states that $H_\mu(\alpha|\mathcal{C})$ increases w.r.t. α and decreases w.r.t. $\mathcal{C}$. Now set

$$H_\mu(\mathcal{U}|\mathcal{C}) = \inf_{\beta \in \mathcal{P}_X, \beta \succeq \mathcal{U}} H_\mu(\beta|\mathcal{C})$$

for $\mathcal{U} \in \mathcal{C}_X$. Clearly, $H_\mu(\mathcal{U}|\mathcal{C})$ increases w.r.t. $\mathcal{U}$ and decreases w.r.t. $\mathcal{C}$.

When $\mu \in \mathcal{M}(X, T)$ and $T^{-1}\mathcal{C} \subseteq \mathcal{C}$ in the sense of μ, it is not hard to see that $H_\mu(\mathcal{U}_0^{n-1}|\mathcal{C})$ is a non-negative and sub-additive sequence for a given $\mathcal{U} \in \mathcal{C}_X$, so we can define

$$h_\mu(T, \mathcal{U}|\mathcal{C}) = \lim_{n \to +\infty} \frac{1}{n} H_\mu(\mathcal{U}_0^{n-1}|\mathcal{C}) = \inf_{n \geq 1} \frac{1}{n} H_\mu(\mathcal{U}_0^{n-1}|\mathcal{C}).$$

Clearly, $h_\mu(T, \mathcal{U}|\mathcal{C})$ also increases w.r.t. $\mathcal{U}$ and decreases w.r.t. $\mathcal{C}$. The *relative measure-theoretical μ-entropy of (X, T) relevant to $\mathcal{C}$* is defined by

$$h_\mu(T, X|\mathcal{C}) = \sup_{\alpha \in \mathcal{P}_X} h_\mu(T, \alpha|\mathcal{C}).$$

Following a similar discussion of [15, Lemma 2.3 (1)], one has

$$h_\mu(T, X|\mathcal{C}) = \sup_{\mathcal{U} \in \mathcal{C}_X^o} h_\mu(T, \mathcal{U}|\mathcal{C}).$$

So, if $\{\mathcal{U}_n\}_{n \in \mathbb{N}} \subseteq \mathcal{C}_X^o$ satisfies $\lim_{n \to +\infty} ||\mathcal{U}_n|| = 0$ then $\lim_{n \to +\infty} h_\mu(T, \mathcal{U}_n|\mathcal{C}) = h_\mu(T, X|\mathcal{C})$, as $h_\mu(T, \mathcal{U}|\mathcal{C})$ increases w.r.t. $\mathcal{U}$. It is not hard to see that

$$h_\mu(T, \mathcal{U}|\mathcal{C}) = \frac{1}{n} h_\mu(T^n, \mathcal{U}_0^{n-1}|\mathcal{C}) \text{ for each } n \in \mathbb{N} \text{ and } \mathcal{U} \in \mathcal{C}_X.$$

If $\mathcal{C} = \{\emptyset, X\}$ (mod μ), for simplicity we shall write $H_\mu(\mathcal{U}|\mathcal{C})$, $h_\mu(T,\mathcal{U}|\mathcal{C})$ and $h_\mu(T,X|\mathcal{C})$ by $H_\mu(\mathcal{U})$, $h_\mu(T,\mathcal{U})$ and $h_\mu(T,X)$, respectively.

The following result is a conditional version of Shannon-McMillan-Breiman Theorem. Its proof is completely similar to the proof of Shannon-McMillan-Breiman Theorem (see e.g. [1, Theorem 4.2], [17, Theorem II.1.5], [14]).

Theorem 4.1. *Let (X,T) be a TDS, $\mu \in \mathcal{M}(X,T)$, $\alpha \in \mathcal{P}_X$ and $\mathcal{C} \subseteq \mathcal{B}_\mu$ a T-invariant sub-σ-algebra (i.e. $T^{-1}\mathcal{C} = \mathcal{C}$ in the sense of μ). Then there exists a T-invariant function $f \in L^1(\mu)$ such that $\int_X f(x)d\mu(x) = h_\mu(T,\alpha|\mathcal{C})$ and*

$$\lim_{n\to+\infty} \frac{I_\mu(\alpha_0^{n-1}|\mathcal{C})(x)}{n} = f(x) \text{ for } \mu\text{-a.e. } x \in X \text{ and in } L^1(\mu).$$

Moreover, if μ is ergodic then $f(x) = h_\mu(T,\alpha|\mathcal{C})$ for μ-a.e. $x \in X$.

Let (X,T) be a TDS, $\mu \in \mathcal{M}(X,T)$ and $\mathcal{B}_\mu$ the completion of $\mathcal{B}_X$ under μ. Then $(X,\mathcal{B}_\mu,\mu,T)$ is a Lebesgue system. If $\{\alpha_i\}_{i\in I}$ is a countable family of finite partitions of X, the partition $\alpha = \bigvee_{i\in I}\alpha_i$ is called a *measurable partition*. The sets $A \in \mathcal{B}_\mu$, which are unions of atoms of α, form a sub-σ-algebra of $\mathcal{B}_\mu$ denoted by $\widehat{\alpha}$ or α if there is no ambiguity. Every sub-σ-algebra of $\mathcal{B}_\mu$ coincides with a sub-σ-algebra constructed in this way (mod μ). Given a measurable partition α, put $\alpha^- = \bigvee_{n=1}^{+\infty} T^{-n}\alpha$ and $\alpha^T = \bigvee_{n=-\infty}^{+\infty} T^{-n}\alpha$. Define in the same way $\mathcal{C}^-$ and $\mathcal{C}^T$ if $\mathcal{C}$ is a sub-σ-algebra of $\mathcal{B}_\mu$. Clearly, for a measurable partition α, $\widehat{\alpha^-} = (\widehat{\alpha})^-$ (mod μ) and $\widehat{\alpha^T} = (\widehat{\alpha})^T$ (mod μ).

Let $\mathcal{C}$ be a sub-σ-algebra of $\mathcal{B}_\mu$ and α the measurable partition of X with $\widehat{\alpha} = \mathcal{C}$ (mod μ). μ can be disintegrated over $\mathcal{C}$ as $\mu = \int_X \mu_x d\mu(x)$ where $\mu_x \in \mathcal{M}(X)$ and $\mu_x(\alpha(x)) = 1$ for μ-a.e. $x \in X$. The disintegration is characterized by (4.1) and (4.2) below: for every $f \in L^1(X,\mathcal{B}_X,\mu)$,

$$\begin{aligned} &f \in L^1(X,\mathcal{B}_X,\mu_x) \text{ for } \mu\text{-a.e. } x \in X \text{ and} \\ &\text{the function } x \mapsto \int_X f(y)d\mu_x(y) \text{ is in } L^1(X,\mathcal{C},\mu); \end{aligned} \tag{4.1}$$

$$\mathbb{E}_\mu(f|\mathcal{C})(x) = \int_X f\, d\mu_x \text{ for } \mu\text{-a.e. } x \in X. \tag{4.2}$$

Then, for any $f \in L^1(X,\mathcal{B}_X,\mu)$, the following holds

$$\int_X \left(\int_X f\, d\mu_x \right) d\mu(x) = \int_X f\, d\mu.$$

Define for μ-a.e. $x \in X$ the set $\Gamma_x = \{y \in X : \mu_x = \mu_y\}$. Then $\mu_x(\Gamma_x) = 1$ for μ-a.e. $x \in X$. Hence given any $f \in L^1(X,\mathcal{B}_X,\mu)$, for μ-a.e. $x \in X$, one has

$$\mathbb{E}_\mu(f|\mathcal{C})(y) = \int_X f\, d\mu_y = \int_X f\, d\mu_x = \mathbb{E}_\mu(f|\mathcal{C})(x) \tag{4.3}$$

for μ_x-a.e. $y \in X$. Particularly, if f is $\mathcal{C}$-measurable, then for μ-a.e. $x \in X$, one has

$$f(y) = f(x) \text{ for } \mu_x\text{-a.e. } y \in X. \tag{4.4}$$

Proposition 4.2. *Let (X,T) be a TDS, $\mu \in \mathcal{M}(X,T)$ and $\mathcal{C} \subseteq \mathcal{B}_\mu$ a T-invariant sub-σ-algebra. If $\mu = \int_X \mu_x d\mu(x)$ is the disintegration of μ over $\mathcal{C}$, then*

(1) *Let $\mathcal{U} \in \mathcal{C}_X$ and $\alpha \in \mathcal{P}_X$ such that each element of $\mathcal{U}$ has a non-empty intersection with at most M elements of α ($M \in \mathbb{N}$). If $f^{\mathcal{C}}_{T,\alpha}(x)$ is the function obtained in Theorem 4.1 for T, α and $\mathcal{C}$, then for μ-a.e. $x \in X$,*
$$h^B_{\mathcal{U}}(T, Z_x) \geq f^{\mathcal{C}}_{T,\alpha}(x) - \log M$$
for any $Z_x \in \mathcal{B}_X$ with $\mu_x(Z_x) > 0$. Particularly, if μ is ergodic, then for μ-a.e. $x \in X$,
$$h^B_{\mathcal{U}}(T, Z_x) \geq h_\mu(T, \alpha|\mathcal{C}) - \log M$$
for any $Z_x \in \mathcal{B}_X$ with $\mu_x(Z_x) > 0$.

(2) *If μ is ergodic then, for μ-a.e. $x \in X$, when $Z_x \in \mathcal{B}_X$ with $\mu_x(Z_x) > 0$ one has*
$$h^B(T, Z_x) \geq h_\mu(T, X|\mathcal{C}).$$

Proof. (1) Note that $\lim_{n\to+\infty} \frac{I_\mu(\alpha_0^{n-1}|\mathcal{C})(x)}{n} = f^{\mathcal{C}}_{T,\alpha}(x)$ for μ-a.e. $x \in X$ and $f^{\mathcal{C}}_{T,\alpha}$ is $\mathcal{C}$-measurable, using (4.3) for all 1_B, $B \in \alpha_0^{n-1}$ and (4.4) for $f^{\mathcal{C}}_{T,\alpha}$, there exists $X_\infty \in \mathcal{B}_X$ with $\mu(X_\infty) = 1$ such that for each $x \in X_\infty$, one can find $W_x \in \mathcal{B}_X$ with $\mu_x(W_x) = 1$ and if $y \in W_x$ then

(a). $\lim_{n\to+\infty} \frac{I_\mu(\alpha_0^{n-1}|\mathcal{C})(y)}{n} = f^{\mathcal{C}}_{T,\alpha}(y) = f^{\mathcal{C}}_{T,\alpha}(x)$.

(b). $\mathbb{E}_\mu(1_B|\mathcal{C})(y) = \mathbb{E}_\mu(1_B|\mathcal{C})(x) = \mu_x(B)$ for any $B \in \alpha_0^{n-1}$ and each $n \in \mathbb{N}$.

Moreover, for any $y \in W_x$, where $x \in X_\infty$, one has

$$\begin{aligned} \lim_{n\to+\infty} \frac{-\log \mu_x(\alpha_0^{n-1}(y))}{n} &= \lim_{n\to+\infty} \frac{-\log \mathbb{E}_\mu(1_{\alpha_0^{n-1}(y)}|\mathcal{C})(y)}{n} \\ &= \lim_{n\to+\infty} \frac{I_\mu(\alpha_0^{n-1}|\mathcal{C})(y)}{n} = f^{\mathcal{C}}_{T,\alpha}(x). \end{aligned} \tag{4.5}$$

For a given $x \in X_\infty$ let $Z_x \in \mathcal{B}_X$ with $\mu_x(Z_x) > 0$. Clearly $\mu_x(Z_x \cap W_x) = \mu_x(Z_x) > 0$. For $\delta > 0$ and $\ell \in \mathbb{N}$, we define
$$Z^\ell_x(\delta) = \{y \in Z_x \cap W_x : \mu_x(\alpha_0^{n-1}(y)) \leq e^{-n(f^{\mathcal{C}}_{T,\alpha}(x)-\delta)} \text{ for each } n \geq \ell\}.$$
Then $\bigcup_{\ell=1}^{+\infty} Z^\ell_x(\delta) = Z_x \cap W_x$ by (4.5). Hence there exists $N \in \mathbb{N}$ such that $\mu_x(Z^N_x(\delta)) > 0$. For $y \in Z^N_x(\delta)$, as $\mu_x(\alpha_0^{n-1}(y)) \leq e^{-n(f^{\mathcal{C}}_{T,\alpha}(x)-\delta)}$ for each $n \geq N$, one has
$$\mu_x(\alpha_0^{n-1}(y)) \leq c(y) e^{-n(f^{\mathcal{C}}_{T,\alpha}(x)-\delta)} \text{ for any } n \in \mathbb{N},$$
where $c(y) = \max\{1, \sum_{i=1}^{N-1} e^{i(f^{\mathcal{C}}_{T,\alpha}(x)-\delta)}\} \in (0, +\infty)$. Thus applying Lemma 3.2 to $Z^N_x(\delta)$ we obtain $h^B_{\mathcal{U}}(T, Z^N_x(\delta)) \geq f^{\mathcal{C}}_{T,\alpha}(x) - \delta - \log M$, so $h^B_{\mathcal{U}}(T, Z_x) \geq f^{\mathcal{C}}_{T,\alpha}(x) - \delta - \log M$. Note that the last inequality is true for any $\delta > 0$, one has $h^B_{\mathcal{U}}(T, Z_x) \geq f^{\mathcal{C}}_{T,\alpha}(x) - \log M$.

(2) For $k \in \mathbb{N}$ we take $\mathcal{U}_k \in \mathcal{C}^o_X$ with $||\mathcal{U}_k|| \leq \frac{1}{k}$. Using [4, Lemma 2] for each $n \in \mathbb{N}$ there exists $\alpha_{n,k} \in \mathcal{P}_X$ such that $\alpha_{n,k} \succeq (\mathcal{U}_k)_0^{n-1}$ and at most $n\#\mathcal{U}_k$ elements

of $\alpha_{n,k}$ can have a point in all their closures, here $\#\mathcal{U}_k$ means the cardinality of $\mathcal{U}_k$. It's easy to construct $\mathcal{U}_{n,k} \in \mathcal{C}_X^o$ such that each element of $\mathcal{U}_{n,k}$ has a non-empty intersection with at most $n\#\mathcal{U}_k$ elements of $\alpha_{n,k}$ (see also [4, Lemma 1]).

Now assume that μ is ergodic. Let $f^{\mathcal{C}}_{T^n,\alpha_{n,k}}(x)$ be the function obtained in Theorem 4.1 for T^n, $\alpha_{n,k}$ and $\mathcal{C}$. Then $f^{\mathcal{C}}_{T^n,\alpha_{n,k}}(x)$ is T^n-invariant and $\int_X f^{\mathcal{C}}_{T^n,\alpha_{n,k}}(x)d\mu(x) = h_\mu(T^n,\alpha_{n,k}|\mathcal{C})$. Let $g_n^k(x) = \frac{1}{n}\sum_{i=0}^{n-1} f^{\mathcal{C}}_{T^n,\alpha_{n,k}}(T^i x)$. Then $g_n^k(x)$ is T-invariant, as $f^{\mathcal{C}}_{T^n,\alpha_{n,k}}(x)$ is T^n-invariant. Moreover, since $\mu \in \mathcal{M}^e(X,T)$, $g_n^k(x)$ is constant and

$$\begin{aligned} g_n^k(x) &\equiv \int_X g_n^k(y)d\mu(y) = \frac{1}{n}\sum_{i=0}^{n-1}\int_X f^{\mathcal{C}}_{T^n,\alpha_{n,k}}(T^i y)d\mu(y) \\ &= \frac{1}{n}\sum_{i=0}^{n-1}\int_X f^{\mathcal{C}}_{T^n,\alpha_{n,k}}(y)d\mu(y) = h_\mu(T^n,\alpha_{n,k}|\mathcal{C}) \end{aligned} \tag{4.6}$$

for μ-a.e. $x \in X$. By (1) for μ-a.e. $x \in X$, if $Z_x \in \mathcal{B}_X$ with $\mu_x(Z_x) > 0$ then

$$h^B_{\mathcal{U}_{n,k}}(T^n, Z_x) \geq f^{\mathcal{C}}_{T^n,\alpha_{n,k}}(x) - \log(n\#\mathcal{U}_k) \text{ for each } k \in \mathbb{N},\ n \in \mathbb{N}. \tag{4.7}$$

Moreover, note that $T\mu_x = \mu_{Tx}$ for μ-a.e. $x \in X$, there exists a T-invariant subset $X_1 \subseteq X$ with $\mu(X_1) = 1$ such that $T\mu_x = \mu_{Tx}$ and both (4.6) and (4.7) hold for all $x \in X_1$.

Now for any given $x \in X_1$ let $Z_x \in \mathcal{B}_X$ with $\mu_x(Z_x) > 0$. Then $T^i x \in X_1$ and $\mu_{T^i x}(T^i Z_x) = T^i\mu_x(T^i Z_x) = \mu_x(Z_x) > 0$ for any $i \geq 0$. By (4.7), for each $k \in \mathbb{N}$, $n \in \mathbb{N}$ and $i \geq 0$,

$$h^B(T^n, T^i Z_x) \geq h^B_{\mathcal{U}_{n,k}}(T^n, T^i Z_x) \geq f^{\mathcal{C}}_{T^n,\alpha_{n,k}}(T^i x) - \log(n\#\mathcal{U}_k).$$

For each $n \in \mathbb{N}$, as $h^B(T^n, Z_x) \geq h^B(T^n, T^i Z_x)$ for each $i \geq 0$ (see Proposition 2.3 (3)), we have

$$\begin{aligned} h^B(T^n, Z_x) &\geq \frac{1}{n}\sum_{i=0}^{n-1} h^B(T^n, T^i Z_x) \geq \frac{1}{n}\sum_{i=0}^{n-1}\left(f^{\mathcal{C}}_{T^n,\alpha_{n,k}}(T^i x) - \log(n\#\mathcal{U}_k)\right) \\ &= g_n^k(x) - \log(n\#\mathcal{U}_k) = h_\mu(T^n,\alpha_{n,k}|\mathcal{C}) - \log(n\#\mathcal{U}_k) \end{aligned}$$

for all $k \in \mathbb{N}$. Then using Proposition 2.3 (4) we have

$$\begin{aligned} h^B(T, Z_x) &= \frac{h^B(T^n, Z_x)}{n} \geq \frac{1}{n}(h_\mu(T^n,\alpha_{n,k}|\mathcal{C}) - \log(n\#\mathcal{U}_k)) \\ &\geq \frac{1}{n}(h_\mu(T^n,(\mathcal{U}_k)_0^{n-1}|\mathcal{C}) - \log(n\#\mathcal{U}_k)) = h_\mu(T,\mathcal{U}_k|\mathcal{C}) - \frac{\log(n\#\mathcal{U}_k)}{n} \end{aligned}$$

for each $k \in \mathbb{N}$ and $n \in \mathbb{N}$. Now fixing $k \in \mathbb{N}$ letting $n \to +\infty$ we get $h^B(T, Z_x) \geq h_\mu(T,\mathcal{U}_k|\mathcal{C})$. Finally letting $k \to +\infty$ we have $h^B(T, Z_x) \geq \lim_{k\to+\infty} h_\mu(T,\mathcal{U}_k|\mathcal{C}) = h_\mu(T, X|\mathcal{C})$. This completes the proof of (2) since $\mu(X_1) = 1$. □

The following result is an application of Proposition 4.2.

Lemma 4.3. *Let (X,T) be a TDS, $\mu \in \mathcal{M}^e(X,T)$ and $\mathcal{C} \subseteq \mathcal{B}_\mu$ a T-invariant sub-σ-algebra. If $\mu = \int_X \mu_x d\mu(x)$ is the disintegration of μ over $\mathcal{C}$, then*

(1) *If $\alpha \in \mathcal{P}_X$ then for μ-a.e. $x \in X$, fixing each x, for each $\epsilon \in (0,1)$ there exists a compact subset $Z_x(\alpha, \epsilon)$ of X such that $\mu_x(Z_x(\alpha, \epsilon)) \geq 1 - \epsilon$ and*

$$h_\alpha^B(T, Z_x(\alpha, \epsilon)) = h_\alpha(T, Z_x(\alpha, \epsilon)) = h_\mu(T, \alpha|\mathcal{C}).$$

(2) *For μ-a.e. $x \in X$, fixing each x, for each $\epsilon \in (0,1)$ there exists a compact subset $Z_x(\epsilon)$ of X such that $\mu_x(Z_x(\epsilon)) \geq 1 - \epsilon$ and*

$$h^B(T, Z_x(\epsilon)) = h(T, Z_x(\epsilon)) = h_\mu(T, X|\mathcal{C}).$$

Proof. (1) Let $\alpha \in \mathcal{P}_X$. As $\mu \in \mathcal{M}^e(X,T)$, by (4.5) there exists $X_\infty \in \mathcal{B}_X$ with $\mu(X_\infty) = 1$ such that for each $x \in X_\infty$, one can find $W_x \in \mathcal{B}_X$ with $\mu_x(W_x) = 1$ such that for each $y \in W_x$

$$\lim_{n \to +\infty} -\frac{1}{n} \log \mu_x(\alpha_0^{n-1}(y)) = h_\mu(T, \alpha|\mathcal{C})$$

(for details see the proof of Proposition 4.2 (1)). By Proposition 4.2 (1), w.l.g. we may require

$$h_\alpha^B(T, Z) \geq h_\mu(T, \alpha|\mathcal{C}) \tag{4.8}$$

for any $x \in X_\infty$ and $Z \in \mathcal{B}_X$ with $\mu_x(Z) > 0$ (if necessary we take a subset of X_∞).

Let $x \in X_\infty$. Obviously, for each $y \in W_x$ and $m \in \mathbb{N}$ there exists $n_{y,m} \in \mathbb{N}$ such that if $n \geq n_{y,m}$ then $-\frac{1}{n}\log \mu_x(\alpha_0^{n-1}(y)) \leq h_\mu(T, \alpha|\mathcal{C}) + \frac{1}{m}$, i.e. $\mu_x(\alpha_0^{n-1}(y)) \geq e^{-n(h_\mu(T,\alpha|\mathcal{C}) + \frac{1}{m})}$. So for each $n \in \mathbb{N}$,

$$\mu_x(\alpha_0^{n-1}(y)) \geq e^{-(n+n_{y,m})(h_\mu(T,\alpha|\mathcal{C}) + \frac{1}{m})}.$$

We introduce a μ_x-measurable function by defining for μ_x-a.e. $y \in X$

$$c_m(y) = \inf_{n \in \mathbb{N}} \frac{\mu_x(\alpha_0^{n-1}(y))}{e^{-n(h_\mu(T,\alpha|\mathcal{C}) + \frac{1}{m})}} \geq e^{-n_{y,m}(h_\mu(T,\alpha|\mathcal{C}) + \frac{1}{m})} > 0.$$

Let $Z_k^m = \{y \in W_x : c_m(y) \geq \frac{1}{k}\}$ for each $k \in \mathbb{N}$. Then Z_k^m is μ_x-measurable and $h_\alpha(T, Z_k^m) \leq h_\mu(T, \alpha|\mathcal{C}) + \frac{1}{m}$ by Lemma 3.3. Moreover, as $\lim_{k \to +\infty} \mu_x(Z_k^m) = 1$, for each $\epsilon \in (0,1)$ there exists a compact subset $B_\epsilon^m \subseteq Z_K^m$ for some $K \in \mathbb{N}$ such that $\mu_x(X \setminus B_\epsilon^m) < \frac{\epsilon}{2^m}$ and $h_\alpha(T, B_\epsilon^m) \leq h_\mu(T, \alpha|\mathcal{C}) + \frac{1}{m}$.

For $\epsilon \in (0,1)$, set $Z_x(\alpha, \epsilon) = \bigcap_{m \in \mathbb{N}} B_\epsilon^m$. Then $Z_x(\alpha, \epsilon)$ is a compact subset of X,

$$\mu_x(Z_x(\alpha, \epsilon)) = 1 - \mu\left(\bigcup_{m \in \mathbb{N}} X \setminus B_\epsilon^m\right) \geq 1 - \sum_{m \in \mathbb{N}} \mu(X \setminus B_\epsilon^m) \geq 1 - \epsilon > 0$$

and

$$h_\alpha(T, Z_x(\alpha, \epsilon)) \leq \inf_{m \in \mathbb{N}} h_\alpha(T, B_\epsilon^m) \leq \inf_{m \in \mathbb{N}} \left(h_\mu(T, \alpha|\mathcal{C}) + \frac{1}{m}\right) = h_\mu(T, \alpha|\mathcal{C}).$$

Moreover, using Lemma 3.1 and (4.8) we have

$$h_\alpha^B(T, Z_x(\alpha, \epsilon)) = h_\alpha(T, Z_x(\alpha, \epsilon)) = h_\mu(T, \alpha|\mathcal{C}).$$

(2) Let $\{\mathcal{U}_n\}_{n \in \mathbb{N}} \subseteq \mathcal{C}_X^o$ with $\lim_{n \to +\infty} ||\mathcal{U}_n|| = 0$. For $n \in \mathbb{N}$ we take $\alpha_n \in \mathcal{P}_X$ with $\alpha_n \succeq \mathcal{U}_n$. By (1) there exists a measurable subset X' of X with $\mu(X') = 1$ such that

if $x \in X'$ then for each $\epsilon \in (0,1)$ and $n \in \mathbb{N}$ there exists a compact subset $Z_x(n,\epsilon)$ such that

$$\mu_x(Z_x(n,\epsilon)) \geq 1 - \frac{\epsilon}{2^n} \text{ and } h_{\mathcal{U}_n}(T, Z_x(n,\epsilon)) \leq h_{\alpha_n}(T, Z_x(n,\epsilon)) = h_\mu(T, \alpha_n|\mathcal{C}).$$

By Proposition 4.2 (2), w.l.g. (if necessary we take a subset of X') we may require

$$h^B(T, Z) \geq h_\mu(T, X|\mathcal{C}) \tag{4.9}$$

for any $x \in X'$ and $Z \in \mathcal{B}_X$ with $\mu_x(Z) > 0$.

Let $x \in X'$. For $\epsilon \in (0,1)$, Set $Z_x(\epsilon) = \bigcap_{n\in\mathbb{N}} Z_x(n,\epsilon)$. Then $Z_x(\epsilon)$ is a compact subset of X,

$$\mu_x(Z_x(\epsilon)) = 1 - \mu\left(\bigcup_{n\in\mathbb{N}} X \setminus Z_x(n,\epsilon)\right) \geq 1 - \sum_{n\in\mathbb{N}} \mu(X \setminus Z_x(n,\epsilon)) \geq 1 - \epsilon > 0$$

and

$$\begin{aligned} h(T, Z_x(\epsilon)) &= \sup_{n\in\mathbb{N}} h_{\mathcal{U}_n}(T, Z_x(\epsilon)) \leq \sup_{n\in\mathbb{N}} h_{\mathcal{U}_n}(T, Z_x(n,\epsilon)) \\ &\leq \sup_{n\in\mathbb{N}} h_\mu(T, \alpha_n|\mathcal{C}) \leq h_\mu(T, X|\mathcal{C}). \end{aligned}$$

Moreover, using Lemma 3.1 and (4.9) we have if $x \in X'$ then $h^B(T, Z_x(\epsilon)) = h(T, Z_x(\epsilon)) = h_\mu(T, X|\mathcal{C})$. This finishes the proof of (2) since $\mu(X') = 1$. □

With the above preparations we can obtain the main result of this section.

Theorem 4.4. *Let (X,T) be a TDS with finite entropy. Then for each $0 \leq h \leq h_{top}(T,X)$ there exists a non-empty compact subset K_h of X such that $h^B(T, K_h) = h(T, K_h) = h$. In particular, (X,T) is lowerable.*

Proof. If $h = h_{\text{top}}(T,X)$, it is true for $K_h = X$ by Proposition 2.3 (1). If $h = 0$, it is true for $K_h = \{x\}$ for any $x \in X$. Now we assume $0 < h < h_{\text{top}}(T,X)$. By the variational principle there exists $\mu \in \mathcal{M}^e(X,T)$ with $h < h_\mu(T) \leq h_{\text{top}}(T,X) < +\infty$. It is well known [14, Theorem 15.11] that there exists a T-invariant sub-σ-algebra $\mathcal{C} \subseteq \mathcal{B}_\mu$ such that $h_\mu(T, X|\mathcal{C}) = h$, where $\mathcal{B}_\mu$ is the completion of $\mathcal{B}_X$ under μ. Then the conclusion follows from Lemma 4.3 (2). □

Let (X,T) be a TDS, $\mu \in \mathcal{M}(X,T)$ and $\mathcal{C} \subseteq \mathcal{B}_\mu$ a T-invariant sub-σ-algebra with $\mu = \int_X \mu_x d\mu(x)$ the disintegration of μ over $\mathcal{C}$. For μ-a.e. $x \in X$, we define

$$h^B_{\mathcal{U}}(T, \mu, x) = \inf\{h^B_{\mathcal{U}}(T, Z) : Z \in \mathcal{B}_X \text{ with } \mu_x(Z) = 1\}$$

for any given $\mathcal{U} \in \mathcal{C}_X$ and

$$h^B(T, \mu, x) = \sup_{\mathcal{U}\in\mathcal{C}^o_X} h^B_{\mathcal{U}}(T, \mu, x).$$

The *essential supremum* of a real valued function f defined on a subset of X with μ-full measure is defined by

$$\mu - \sup f(x) = \inf_{\mu(X')=1} \sup_{x\in X'} f(x).$$

We are not sure of the μ-measurability of functions both $h^B_{\mathcal{U}}(T,\mu,x)$ and $h^B(T,\mu,x)$ w.r.t. $x \in X$. Whereas, using Proposition 4.2 we have the following result.

Corollary 4.5. *Let (X,T) be a TDS, $\mu \in \mathcal{M}(X,T)$ and $\mathcal{C} \subseteq \mathcal{B}_\mu$ a T-invariant sub-σ-algebra. If $\mu = \int_X \mu_x d\mu(x)$ is the disintegration of μ over $\mathcal{C}$, then*

(1) *Let $\mathcal{U} \in \mathcal{C}_X$ and $\alpha \in \mathcal{P}_X$ with $f^{\mathcal{C}}_{T,\alpha}(x)$ the function obtained in Theorem 4.1 for T, $\alpha \in \mathcal{P}_X$ and $\mathcal{C}$. Assume that each element of $\mathcal{U}$ has a non-empty intersection with at most M elements of α ($M \in \mathbb{N}$). Then for μ-a.e. $x \in X$,*
$$h^B_{\mathcal{U}}(T,\mu,x) \geq f^{\mathcal{C}}_{T,\alpha}(x) - \log M$$
and if μ is ergodic then
$$h^B_{\mathcal{U}}(T,\mu,x) \geq h_\mu(T,\alpha|\mathcal{C}) - \log M.$$

(2) *$\mu-\sup h^B(T,\mu,x) \geq h_\mu(T,X|\mathcal{C})$; moreover, if μ is ergodic then $h^B(T,\mu,x) = h_\mu(T,X|\mathcal{C})$ for μ-a.e. $x \in X$.*

Proof. (1) is just a direct corollary of Proposition 4.2 (1).

(2) For $k \in \mathbb{N}$ we take $\mathcal{U}_k \in \mathcal{C}^o_X$ with $||\mathcal{U}_k|| \leq \frac{1}{k}$. Then, for each $n \in \mathbb{N}$, we take $\alpha_{n,k} \in \mathcal{P}_X$ such that $\alpha_{n,k} \succeq (\mathcal{U}_k)_0^{n-1}$ and at most $n\#\mathcal{U}_k$ elements of $\alpha_{n,k}$ can have a point in all their closures, and take $\mathcal{U}_{n,k} \in \mathcal{C}^o_X$ such that each element of $\mathcal{U}_{n,k}$ has a non-empty intersection with at most $n\#\mathcal{U}_k$ elements of $\alpha_{n,k}$.

Let $f^{\mathcal{C}}_{T^n,\alpha_{n,k}}(x)$ be the function obtained in Theorem 4.1 for T^n, $\alpha_{n,k}$ and $\mathcal{C}$. Then $f^{\mathcal{C}}_{T^n,\alpha_{n,k}}(x)$ is T^n-invariant and $\int_X f^{\mathcal{C}}_{T^n,\alpha_{n,k}}(x) d\mu(x) = h_\mu(T^n,\alpha_{n,k}|\mathcal{C})$. Then using (1), for μ-a.e. $x \in X$,
$$\begin{aligned} & nh^B_{\mathcal{U}_{n,k}}(T,\mu,x) = \inf\{nh^B_{\mathcal{U}_{n,k}}(T,Z) : Z \in \mathcal{B}_X \text{ with } \mu_x(Z) = 1\} \\ \geq\ & \inf\{h^B_{\mathcal{U}_{n,k}}(T^n,Z) : Z \in \mathcal{B}_X \text{ with } \mu_x(Z) = 1\} \text{ (by Proposition 2.3 (4))} \\ \geq\ & f^{\mathcal{C}}_{T^n,\alpha_{n,k}}(x) - \log(n\#\mathcal{U}_k) \text{ (using (1))}. \end{aligned}$$
Hence,
$$\begin{aligned} & \mu - \sup h^B(T,\mu,x) \geq \mu - \sup h^B_{\mathcal{U}_{n,k}}(T,\mu,x) \\ \geq\ & \frac{1}{n}\int_X \left(f^{\mathcal{C}}_{T^n,\alpha_{n,k}}(x) - \log(n\#\mathcal{U}_k)\right) d\mu(x) = \frac{1}{n}\left(h_\mu(T^n,\alpha_{n,k}|\mathcal{C}) - \log(n\#\mathcal{U}_k)\right) \\ \geq\ & \frac{1}{n}(h_\mu(T^n,(\mathcal{U}_k)_0^{n-1}|\mathcal{C}) - \log(n\#\mathcal{U}_k)) = h_\mu(T,\mathcal{U}_k|\mathcal{C}) - \frac{1}{n}\log(n\#\mathcal{U}_k). \end{aligned}$$
Fixing $k \in \mathbb{N}$ letting $n \to +\infty$ in the above inequality we obtain $\mu-\sup h^B(T,\mu,x) \geq h_\mu(T,\mathcal{U}_k|\mathcal{C})$. Then letting $k \to +\infty$ we obtain $\mu - \sup h^B(T,\mu,x) \geq h_\mu(T,X|\mathcal{C})$.

Now we assume that μ is ergodic. First, by Proposition 4.2 (2), we know $h^B(T,\mu,x) \geq h_\mu(T,X|\mathcal{C})$ for μ-a.e. $x \in X$. Secondly using Lemma 4.3 (2), there exists a measurable subset X' of X with $\mu(X') = 1$ such that if $x \in X'$ then for each $\ell \in \mathbb{N}$ there exists a compact subset $Z_x(\ell)$ of X such that $\mu_x(Z_x(\epsilon)) \geq 1 - \frac{1}{2^\ell}$ and $h^B(T,Z_x(\ell)) = h_\mu(T,X|\mathcal{C})$. Next for each $x \in X'$, let $Z_x = \bigcup_{\ell \in \mathbb{N}} Z_x(\ell)$. Then $Z_x \in \mathcal{B}_X$ with $\mu(Z_x) = 1$ and $h^B(T,Z_x) = \sup_{\ell \in \mathbb{N}} h^B(T,Z_x(\ell)) = h_\mu(T,X|\mathcal{C})$. This

implies $h^B(T,\mu,x) \leq h_\mu(T,X|\mathcal{C})$. Collecting terms, $h^B(T,\mu,x) = h_\mu(T,X|\mathcal{C})$ for μ-a.e. $x \in X$. $\square$

Following from the proof of Corollary 4.5, we are easy to show the following result.

Corollary 4.6. *Let (X,T) be a TDS, $\mu \in \mathcal{M}^e(X,T)$ and $\mathcal{C} \subseteq \mathcal{B}_\mu$ a T-invariant sub-σ-algebra. If $\mu = \int_X \mu_x d\mu(x)$ is the disintegration of μ over $\mathcal{C}$, then*

1. *If $\alpha \in \mathcal{P}_X$ then for μ-a.e. $x \in X$ there exists $Z_x \in \mathcal{B}_X$ such that $\mu_x(Z_x) = 1$ and $h^B_\alpha(T,Z_x) = h_\mu(T,\alpha|\mathcal{C})$. Moreover, $h^B_\alpha(T,\mu,x) = h_\mu(T,\alpha|\mathcal{C})$ for μ-a.e. $x \in X$.*
2. *For μ-a.e. $x \in X$ there exists $Z_x \in \mathcal{B}_X$ such that $\mu_x(Z_x) = 1$ and $h^B(T,Z_x) = h_\mu(T,X|\mathcal{C})$.*

Remark 4.7. *We can't expect similar results hold for topological entropy of subsets using open covers. For example, let (X,T) be a minimal TDS, $\mu \in \mathcal{M}^e(X,T)$ and $\mathcal{C} = \{\emptyset, X\}$ such that $0 < h_\mu(T,X) < h_{top}(T,X)$. Let $\mu = \int_X \mu_x d\mu(x)$ be the disintegration of μ over $\mathcal{C}$, then $\mu_x = \mu$ for μ-a.e. $x \in X$. Thus for μ-a.e. $x \in X$, if $Z \in \mathcal{B}_X$ with $\mu_x(Z) = 1$ then $\overline{Z} = X$, which implies $h(T,Z) = h(T,\overline{Z}) = h_{top}(T,X) > h_\mu(T,X) = h_\mu(T,X|\mathcal{C})$.*

5. Expansive cases

In this section by direct construction we shall prove that each expansive TDS is **HUL**. Recall that we say a TDS (X,T) is *expansive* if there exists $\delta > 0$ such that $x \neq y$ implies $\sup_{n\in\mathbb{Z}} d(T^n x, T^n y) > \delta$. In this case, δ is called an *expansive constant*. In particular, each symbolic TDS is expansive.

To do this let's first recall [30, Remark 5.13]. Let (X,T) be a TDS with metric d and E a compact subset. For each $\epsilon > 0$ and $x \in E$ we define

$$h_d(x,\epsilon,E) = \inf\{r(d,T,\epsilon,K) : K \text{ is a compact neighborhood of } x \text{ in } E\}.$$

Let $h(x,E) = \lim_{\epsilon\to 0+} h_d(x,\epsilon,E)$. Its value depends only on the topology on X. The following is [30, Remark 5.13].

Theorem 5.1. *Let (X,T) be a TDS with metric d and E a compact subset. Then*

1. *$h_d(x,\epsilon,E)$ is u.s.c. on E and $\sup_{x\in E} h(x,E) = h(T,E)$.*
2. *For each $x \in E$ there is a countable compact subset $E_x \subseteq E$ with a unique limit point x such that $h(T,E_x) = h(x,E)$.*
3. *There is a countable compact subset $E' \subseteq E$ with $h(T,E') = h(T,E)$. Moreover, E' can be chosen such that the set of its limit points has at most one limit point, and E' has a unique limit point iff there is $x \in E$ with $h(x,E) = h(T,E)$.*

The first result is the following lemma.

Lemma 5.2. *Let (X,T) be a TDS with metric d and $K \subseteq X$ a compact subset with $h(T,K) > 0$. Then for any $0 < h < h(T,K)$ there is a $\delta_0 > 0$ such that if $0 < \delta \leq \delta_0$ then there is a countable compact subset $K_{h,\delta} \subseteq K$ with a unique limit point such that $s(d,T,\delta,K_{h,\delta}) = h$.*

Proof. Let $0 < h < h(T, K)$. By Theorem 5.1 there exists a countable compact subset $K_0 \subseteq K$ with a unique limit point x_0 such that $h(T, K_0) > h$, thus for some $\delta_0 > 0$ if $0 < \delta \leq \delta_0$ then $s(d, T, \delta, K_0) > h$. Now let $0 < \delta \leq \delta_0$ be fixed.

Define l_1 to be the minimal integer $n \in \mathbb{N}$ such that

$$\exists B_1 \subseteq K_0 \text{ is } (n, \delta)\text{-separated w.r.t. } T \text{ s.t. } |B_1| = [e^{nh}] + 2,$$

here $|B_1|$ means the cardinality of B_1. It is clear that l_1 is finite, as $s(d, T, \delta, K_0) > h$. Let $A_1 = D_1 \subseteq K_0$ be (l_1, δ)-separated w.r.t. T with $|A_1| = [e^{l_1 h}] + 1$ and $x_0 \notin A_1$.

Define l_2 to be the minimal integer $n > l_1$ such that

$$\exists B_2 \subseteq (B_{d_{l_1}}(x_0, \delta) \cap K_0) \setminus A_1 \text{ is } (n, \delta)\text{-separated w.r.t. } T \text{ s.t. } |B_2| = [e^{nh}] - [e^{l_1 h}] + 2,$$

where $B_{d_{l_1}}(x_0, \delta)$ denotes the open ball with center x_0 and radius δ (in the sense of d_{l_1}-metric). Since x_0 is the unique limit point of the countable compact subset $K_0 \subseteq K$, $K_0 \setminus ((B_{d_{l_1}}(x_0, \delta) \cap K_0) \setminus A_1)$ is a finite subset, so

$$s(d, T, \delta, (B_{d_{l_1}}(x_0, \delta) \cap K_0) \setminus A_1) = s(d, T, \delta, K_0) > h,$$

which implies that $l_2 > l_1$ is finite. Let $D_2 \subseteq (B_{d_{l_1}}(x_0, \delta) \cap K_0) \setminus A_1$ be (l_2, δ)-separated w.r.t. T with $|D_2| = [e^{l_2 h}] - [e^{l_1 h}] + 1$ and $x_0 \notin D_2$. Set $A_2 = A_1 \cup D_2 \not\ni x_0$. Then $|A_2| = [e^{l_2 h}] + 2$ and $A_2 \subseteq K_0$ is (l_2, δ)-separated w.r.t. T.

By induction there are $l_1 < l_2 < \cdots$ and $A_1 \subseteq A_2 \subseteq \cdots \subseteq K_0$ such that for each $i \in \mathbb{N}$

(1) $x_0 \notin A_i$ and $A_{i+1} \setminus A_i \subseteq B_{d_{l_i}}(x_0, \delta) \cap K_0$.

(2) A_i is (l_i, δ)-separated w.r.t. T and $|A_i| = [e^{l_i h}] + i$.

Set $A_\infty = \{x_0\} \cup \bigcup_{i \geq 1} A_i \subseteq K_0$. Then $A_\infty (\subseteq K_0) \subseteq K$ is a countable compact subset with x_0 as its unique limit point in X. If $l_n \leq l < l_{n+1}$ then let $A \subseteq A_\infty$ be (l, δ)-separated w.r.t. T. As $A \setminus A_n \subseteq (B_{d_{l_n}}(x_0, \delta) \cap K_0) \setminus A_n$ is (l, δ)-separated w.r.t. T, because of the definition of l_{n+1} we have $|A \setminus A_n| \leq [e^{lh}] - [e^{l_n h}] + 1$, which implies

$$|A| \leq |A \setminus A_n| + |A_n| \leq ([e^{lh}] - [e^{l_n h}] + 1) + ([e^{l_n h}] + n) = [e^{lh}] + n + 1.$$

Then $s_l(d, T, \delta, A_\infty) \leq [e^{lh}] + n + 1$ for all $l_n \leq l < l_{n+1}$. Note that $s_{l_n}(d, T, \delta, A_\infty) \geq [e^{l_n h}] + n$, we conclude $s(d, T, \delta, A_\infty) = h$. Take $K_{h,\delta} = A_\infty$. This completes the proof. □

Before proving that each expansive TDS is **HUL** we need the following result.

Lemma 5.3. *Let (X, T) be an expansive TDS with metric d and an expansive constant $\delta > 0$. Then for any compact subset $K \subseteq X$, $h(T, K) = s(d, T, \frac{\delta}{2}, K)$.*

Proof. Let $K \subseteq X$ be a compact subset and $\epsilon > 0$. We claim that there exists $n(\epsilon) \in \mathbb{N}$ such that for $x, y \in X$, $d^*_{n(\epsilon)}(x, y) \leq \frac{\delta}{2}$ implies $d(x, y) \leq \epsilon$, where $d^*_{n(\epsilon)}(x, y) = \max_{i=-n(\epsilon)}^{n(\epsilon)} d(T^i x, T^i y)$. In fact, if it is not the case, then for each $n \in \mathbb{N}$ there exist $x_n, y_n \in X$ such that $d^*_n(x_n, y_n) \leq \frac{\delta}{2}$ and $d(x_n, y_n) > \epsilon$. W.l.g. we assume $\lim_{n \to +\infty}(x_n, y_n) = (x, y)$. Then $d(x, y) \geq \epsilon$ and $d^*_m(x, y) \leq \frac{\delta}{2}$ for each $m \in \mathbb{N}$, which contradicts that δ is an expansive constant of (X, T).

Now for each $m \in \mathbb{N}$, let E be an (m, ϵ)-separated subset of K w.r.t. T, then $T^{-n(\epsilon)}E$ is an $(m + 2n(\epsilon), \frac{\delta}{2})$-separated subset of $T^{-n(\epsilon)}K$ w.r.t. T. Hence

$$\begin{aligned} s(d,T,\epsilon,K) &= \limsup_{m\to+\infty} \frac{1}{m} \log s_m(d,T,\epsilon,K) \\ &\le \limsup_{m\to+\infty} \frac{1}{m} \log s_{m+2n(\epsilon)} \left(d, T, \frac{\delta}{2}, T^{-n(\epsilon)}K\right) \\ &= s\left(d, T, \frac{\delta}{2}, T^{-n(\epsilon)}K\right) = s\left(d, T, \frac{\delta}{2}, K\right). \end{aligned}$$

Sine $\epsilon > 0$ is arbitrary, letting $\epsilon \to 0+$ we conclude $h(T, K) = s(d, T, \frac{\delta}{2}, K)$. □

Now we are ready to prove the main result in this section.

Theorem 5.4. *Each expansive TDS is* **HUL**.

Proof. Let (X, T) be an expansive TDS with metric d and an expansive constant $2\delta > 0$.

Let $E \subseteq X$ be a non-empty compact subset and $0 \le h \le h(T, E)$. If $h = 0$ then $h(T, \{x\}) = 0$ for any $x \in E$. Now assume $h = h(T, E) > 0$, then by Theorem 5.1 (1), $h_d(x, \delta, E)$ is u.s.c. on E and $h(T, E) = \sup_{x\in X} h(x, E)$. Note that

$$\begin{aligned} h(x,E) &= \lim_{\epsilon\to 0+} \inf\{r(d,T,\epsilon,K) : K \text{ is a compact neighborhood of } x \text{ in } E\} \\ &= \lim_{\epsilon\to 0+} \inf\{s(d,T,\epsilon,K) : K \text{ is a compact neighborhood of } x \text{ in } E\} \\ &= h_d(x,\delta,E) \text{ (using Lemma 5.3)}, \end{aligned}$$

then $h(T, E) = \max_{x\in E} h_d(x, \delta, E)$. Say $x_0 \in E$ with $h_d(x_0, \delta, E)(= h(x_0, E)) = h(T, E) > 0$. By Theorem 5.1 (2) there exists a countable compact subset $K_0 \subseteq E$ with x_0 as its unique limit point such that $h(T, K_0) = h(x_0, E) = h(T, E)$. This completes the proof in the case of $h = h(T, E) > 0$.

Now assume $0 < h < h(T, E)$. By Lemma 5.2 there exist $0 < \epsilon \le \delta$ small enough and a countable compact subset $K_h \subseteq E$ with a unique limit point such that $s(d, T, \epsilon, K_h) = h$. Since 2ϵ is also an expansive constant, $h(T, K_h) = h$ by Lemma 5.3.

Thus for each non-empty compact subset E and each $0 \le h \le h(T, E)$ there is a non-empty compact subset $K_h \subseteq K$ with a unique limit point such that $h(T, K_h) = h$, that is, TDS (X, T) is **HUL**. □

Note that we can also introduce the **HUL** property for a GTDS, the results of this section remain true for a GTDS. In particular, following similar discussions, the results in Lemma 5.3 and Theorem 5.4 also hold for each positively expansive dynamical system. Let X be a compact metric space endowed with a continuous surjection $T : X \to X$ and a compatible metric d. Recall that we say (X, T) *positively expansive* if there exists $\delta > 0$ such that $x \ne y$ implies $\sup_{n\in\mathbb{Z}_+} d(T^n x, T^n y) > \delta$ (see for example [27]). We also call δ an *expansive constant*.

6. A **HUL** TDS is asymptotically h-expansive

In this section we shall answer Question 1.4 partially. Note that the invertibility can be removed for TDSs considered in this section without changing our results.

We discuss two classes of weak expansiveness: the h-expansiveness and asymptotical h-expansiveness, introduced by Bowen [3] and Misiurewicz [24], respectively. Let (X,T) be a GTDS with metric d. For each $\epsilon > 0$ we define

$$h_T^*(\epsilon) = \sup_{x \in X} h(T, \Phi_\epsilon(x)),\ \text{where } \Phi_\epsilon(x) = \{y \in X : d(T^n x, T^n y) \leq \epsilon \text{ if } n \geq 0\}.$$

(X,T) is called *h-expansive* if there exists an $\epsilon > 0$ such that $h_T^*(\epsilon) = 0$, and is called *asymptotically h-expansive* if $\lim_{\epsilon \to 0+} h_T^*(\epsilon) = 0$. It is shown by Bowen [3] that positively expansive systems, expansive homeomorphisms, endomorphisms of a compact Lie group and Axiom A diffeomorphisms are all h-expansive, by Misiurewicz [25] that every continuous endomorphism of a compact metric group is asymptotically h-expansive if it has finite entropy, and by Buzzi [7] that any C^∞ diffeomorphism on a compact manifold is asymptotically h-expansive.

In this section we prove that each **HUL** TDS is asymptotically h-expansive.

The following two results seem too technical but interesting themselves, which are needed in proving the main result of this section.

Theorem 6.1. *Let (X,T) be a TDS. Then for any compact subset $K \subseteq X$ with $h(T,K) > 0$, there is a countable infinite compact subset $K_\infty \subseteq K$ such that $h(T, K_\infty) = 0$.*

Proof. First, there is a countable compact subset $K_0 = \{x, x_1, x_2, \cdots\} \subseteq K$ such that $h = h(T, K_0) > 0$ and $\lim_{n \to +\infty} x_n = x$ (using Theorem 5.1).

Let d be a metric on (X,T). For sufficiently small $\epsilon_1 > 0$ let $K_1 \subseteq K_0$ be the subset constructed in Lemma 5.2 such that $s(d, T, \epsilon_1, K_1) = \frac{h}{2}$. Now if K_n, $n \in \mathbb{N}$, is constructed, for a more smaller $0 < \epsilon_{n+1} < \epsilon_n$, by Lemma 5.2 we let K_{n+1} be a proper compact subset of K_n with $s(d, T, \epsilon_{n+1}, K_{n+1}) = \frac{h}{n+2}$. In fact, we can require that $\lim_{n \to +\infty} \epsilon_n = 0$. Now let $K_\infty = \{x, y_1, y_2, \cdots\}$ be a subset of K_0, where $y_n \in K_n \setminus K_{n+1}$ for each $n \in \mathbb{N}$.

It is clear that $K_\infty \subseteq K$ is a countable infinite compact subset, and

$$s(d, T, \epsilon_n, K_\infty) \leq s(d, T, \epsilon_n, K_n) \leq \frac{h}{n}$$

for each $n \in \mathbb{N}$, as $K_\infty \setminus \{y_1, \cdots, y_{n-1}\} \subseteq K_n$. Hence we have

$$h(T, K_\infty) = \lim_{n \to +\infty} s(d, T, \epsilon_n, K_\infty) = 0.$$

This completes the proof. □

Lemma 6.2. *Let (X,T) be a TDS. Assume that $\{B_n\}_{n \in \mathbb{N}} \subseteq 2^X$ satisfies $\lim_{n \to +\infty} B_n = \{x_0\}$ (in the sense of Hausdorff metric) for some $x_0 \in X$ and*

$$\inf_{J \in \mathbb{Z}_+} \lim_{n \to +\infty} \sup_{j \geq J} diam(T^j B_n) = 0. \tag{6.1}$$

Let $x_n \in B_n$ for each $n \in \mathbb{N}$. Then

$$h\left(T, \bigcup_{n=1}^{+\infty} B_n \cup \{x_0\}\right) = \max\left\{\sup_{n\in\mathbb{N}} h(T, B_n), h(T, \{x_i\}_0^\infty)\right\}. \tag{6.2}$$

If in addition $x_0 \notin B_n$ for each $n \in \mathbb{N}$, then any countable compact subset of $\bigcup_{n=1}^{+\infty} B_n \cup \{x_0\}$ with a unique limit point x_0 has entropy at most $h(T, \{x_i\}_0^\infty)$.

Proof. Let d be a metric on (X, T) and $\epsilon > 0$. Thus by (6.1) there exist $i_\epsilon, j_\epsilon \in \mathbb{N}$ such that if $i \geq i_\epsilon$ then $\epsilon_i < \frac{\epsilon}{4}$, where $\epsilon_i = \sup_{j\geq j_\epsilon} \operatorname{diam}(T^j B_i)$. Set $X_1 = T^{j_\epsilon}(\bigcup_{i\in\mathbb{N}} B_i \cup \{x_0\})$.

For $n \in \mathbb{N}$, let E_n be an (n, ϵ)-separated subset of X_1 w.r.t. T with $s_n(d, T, \epsilon, X_1) = |E_n|$. It is clear that if $i \geq i_\epsilon$ then $|E_n \cap T^{j_\epsilon} B_i| \leq 1$, and if $y \in E_n \cap T^{j_\epsilon} B_i$ then $d_n(y, T^{j_\epsilon} x_i) \leq \epsilon_i < \frac{\epsilon}{4}$. Say $E_n \cap T^{j_\epsilon}(\bigcup_{i\geq i_\epsilon} B_i) = \{y_1, \cdots, y_l\}$. For each $1 \leq r \leq l$, there exists $i_r \geq i_\epsilon$ such that $y_r \in E_n \cap T^{j_\epsilon} B_{i_r}$. Let $F_n = \{T^{j_\epsilon} x_{i_1}, \cdots, T^{j_\epsilon} x_{i_r}\}$. For each $1 \leq r_1 < r_2 \leq l$,

$$d_n(T^{j_\epsilon} x_{i_{r_1}}, T^{j_\epsilon} x_{i_{r_2}}) \geq d_n(y_{r_1}, y_{r_2}) - (d_n(y_{r_1}, T^{j_\epsilon} x_{i_{r_1}}) + d_n(y_{r_2}, T^{j_\epsilon} x_{i_{r_2}})) > \frac{\epsilon}{2}.$$

Hence, F_n is an $(n, \frac{\epsilon}{2})$-separated subset of $T^{j_\epsilon}(\{x_i\}_0^\infty)$ w.r.t. T, which implies $s_n(d, T, \frac{\epsilon}{2}, T^{j_\epsilon}(\{x_i\}_0^\infty)) \geq l$. Note that $G_n \doteq E_n \cap T^{j_\epsilon}(\bigcup_{i=1}^{i_\epsilon - 1} B_i)$ is an (n, ϵ)-separated subset of $T^{j_\epsilon}(\bigcup_{i=1}^{i_\epsilon-1} B_i)$ w.r.t. T, we have

$$\begin{aligned} s_n(d, T, \epsilon, X_1) &= |E_n| \leq |G_n \cup \{y_1, \cdots, y_l\} \cup \{T^{j_\epsilon} x_0\}| \leq |G_n| + l + 1 \\ &\leq s_n\left(d, T, \epsilon, T^{j_\epsilon}\left(\bigcup_{i=1}^{i_\epsilon-1} B_i\right)\right) + s_n\left(d, T, \frac{\epsilon}{2}, T^{j_\epsilon}(\{x_i\}_0^\infty)\right) + 1, \end{aligned}$$

which implies that

$$\begin{aligned} & s\left(d, T, \epsilon, \bigcup_{i\in\mathbb{N}} B_i \cup \{x_0\}\right) = s(d, T, \epsilon, X_1) \\ \leq\ & \max\left\{ s\left(d, T, \epsilon, T^{j_\epsilon}\left(\bigcup_{j=1}^{i_\epsilon-1} B_j\right)\right), s\left(d, T, \frac{\epsilon}{2}, T^{j_\epsilon}(\{x_i\}_0^\infty)\right)\right\} \\ =\ & \max\left\{ s\left(d, T, \epsilon, \bigcup_{j=1}^{i_\epsilon-1} B_j\right), s\left(d, T, \frac{\epsilon}{2}, \{x_i\}_0^\infty\right)\right\} \\ \leq\ & \max\left\{ \max_{1\leq j\leq i_\epsilon-1} h(T, B_i), h(T, \{x_i\}_0^\infty)\right\}. \end{aligned}$$

Thus we obtain the direction "$\leq$" of (6.2). The other direction is obvious.

If in additional $x_0 \notin B_n$ for each $n \in \mathbb{N}$, let $K \subseteq \bigcup_{n=1}^{+\infty} B_n \cup \{x_0\}$ be any countable compact subset with a unique limit point x_0. Set $B_n' = K \cap B_n$. Then B_n' is finite for each $n \in \mathbb{N}$, as x_0 is the unique limit point of K and $x_0 \notin B_n'$. So we have

$\{B'_n \cup \{x_n\}\}_{n\in\mathbb{N}} \subseteq 2^X$ and

$$\begin{aligned} h(T,K) &\le h(T, K\cup\{x_i\}_{i=0}^{\infty}) = h\left(T, \bigcup_{n=1}^{+\infty}(B'_n \cup \{x_n\}) \cup \{x_0\}\right) \\ &= \max\left\{\sup_{n\in\mathbb{N}} h(T, B'_n\cup\{x_n\}), h(T,\{x_i\}_0^{\infty})\right\} \text{ (using (6.2))} \\ &= h(T,\{x_i\}_0^{\infty}) \text{ (as } B'_n\cup\{x_n\} \text{ is a finite subset for each } n\in\mathbb{N}). \end{aligned}$$

This finishes the proof. □

Remark 6.3. *Without the assumption of* (6.1)*, in general Lemma 6.2 doesn't hold.*

For example, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of X with limit $x\in X$ and $h(T,\{x,x_1,x_2,\cdots\}) = a > 0$.

By Theorem 6.1 there is a sub-sequence $\{n_i\}_{i\in\mathbb{N}}$ such that $h(T,\{x,x_{n_1},x_{n_2},\cdots\}) = 0$. Let $B_j = \{x_{n_{j-1}}, x_{n_{j-1}+1},\cdots, x_{n_j-1}\}$ for each $j\in\mathbb{N}$, where $n_0 = 1$. Then $\lim_{j\to+\infty}\mathrm{diam}(B_j) = 0$ and $h(T,\bigcup_{j\in\mathbb{N}} B_j\cup\{x\}) = a > 0$, but $\sup_{j\in\mathbb{N}} h(T,B_j) + h(T,\{x,x_{n_1},x_{n_2},\cdots\}) = 0$.

In fact, for a good choice of the sub-sequence $\{n_i\}_{i\in\mathbb{N}}$ in the above construction, we can require that $h(T,\bigcup_{i\in\mathbb{N}} B_{k_i}\cup\{x\}) = a > 0$ for any sub-sequence $\{k_i\}_{i\in\mathbb{N}}\subseteq\mathbb{N}$. This is done as follows.

For each $j\in\mathbb{N}$ we can select a sub-sequence $\{m_k^j\}_{k\in\mathbb{N}}\subseteq\mathbb{N}$ such that

$$\begin{aligned} &\lim_{k\to+\infty}\frac{\log s_{m_k^j}(d,T,\frac{1}{j},\{x_l : l\ge n_i\})}{m_k^j} \\ = \ & s\left(d,T,\frac{1}{j},\{x_l : l\ge n_i\}\right)\left(= s\left(d,T,\frac{1}{j},\{x,x_1,x_2,\ldots\}\right)\right) \end{aligned}$$

for each $i\in\mathbb{N}$. We may assume (replace the sequences $\{n_i\}_{i\in\mathbb{N}}$ and $\{m_k^j\}_{k\in\mathbb{N}}$ by sub-sequences if necessary)

$$s_{m_i^j}\left(d,T,\frac{1}{j},\{x_l : n_i\le l< n_{i+1}\}\right) \ge e^{m_i^j(s(d,T,\frac{1}{j},\{x,x_1,x_2,\cdots\})-\frac{1}{i})} \text{ if } 1\le j\le i.$$

Let $B_j = \{x_{n_{j-1}}, x_{n_{j-1}+1},\cdots,x_{n_j-1}\}$. Then

$$\sup_{j\in\mathbb{N}} h(T,B_j) + h(T,\{x,x_{n_1},x_{n_2},\cdots\}) = 0.$$

Now for any sub-sequence $\{k_i\}_{i\in\mathbb{N}}\subseteq\mathbb{N}$ we have: if $l\in\mathbb{N}$ and $1\le j\le k_l$ then

$$s_{m_{k_l}^j}\left(d,T,\frac{1}{j},\bigcup_{i\in\mathbb{N}} B_{k_i}\cup\{x\}\right) \ge s_{m_{k_l}^j}\left(d,T,\frac{1}{j},B_{k_l}\right) \ge e^{m_{k_l}^j(s(d,T,\frac{1}{j},\{x,x_1,x_2,\cdots\})-\frac{1}{k_l})},$$

which implies that for each fixed $j \in \mathbb{N}$

$$s\left(d, T, \frac{1}{j}, \{x, x_1, x_2, \cdots\}\right) \geq s\left(d, T, \frac{1}{j}, \bigcup_{i \in \mathbb{N}} B_{k_i} \cup \{x\}\right)$$

$$\geq \limsup_{l \to +\infty} \frac{1}{m^j_{k_l}} \log s_{m^j_{k_l}}\left(d, T, \frac{1}{j}, \bigcup_{i \in \mathbb{N}} B_{k_i} \cup \{x\}\right) \geq s\left(d, T, \frac{1}{j}, \{x, x_1, x_2, \cdots\}\right).$$

Then letting $j \to +\infty$ we have $h(T, \bigcup_{i \in \mathbb{N}} B_{k_i} \cup \{x\}) = h(T, \{x, x_1, x_2, \cdots\}) = a$.

Now we are ready to prove the main result in this section.

Theorem 6.4. *Each* **HUL** *TDS is asymptotically h-expansive.*

Proof. Let (X, T) be a **HUL** TDS with metric d. Assume the contrary that (X, T) is not asymptotically h-expansive, i.e. $a = h^*(T) = \lim_{\epsilon \to 0+} h^*_T(\epsilon) > 0$. Then there exist a sequence $\{x_i\}_{i \in \mathbb{N}} \subseteq X$ with limit x and a sequence $\{\epsilon_i\}_{i \in \mathbb{N}}$ of positive numbers with limit 0 such that $\lim_{i \to +\infty} h(T, \Phi_{\epsilon_i}(x_i)) = a$. There are two cases.

Case 1. There exist infinitely many $i \in \mathbb{N}$ such that for which $x \notin \Phi_{\epsilon_i}(x_i)$. Thus w.l.g. we assume $x \notin \Phi_{\epsilon_i}(x_i)$ for each $i \in \mathbb{N}$.

Since (X, T) is **HUL**, for each $i \in \mathbb{N}$ we can take a countable infinite compact subset $X_i \subseteq \Phi_{\epsilon_i}(x_i)$ with a unique limit point y_i such that $a_i \doteq h(T, X_i) < a$ and $\lim_{i \to +\infty} a_i = a$. Clearly, $\lim_{i \to +\infty} y_i = x$. Moreover, by Theorem 6.1 we may assume that $h(T, \{x, y_1, y_2, \cdots\}) = 0$ (if necessary we take a sub-sequence).

By Lemma 6.2, $h(T, \bigcup_{i \in \mathbb{N}} X_i \cup \{x\}) = \max\{\sup_{i \in \mathbb{N}} h(T, X_i), h(T, \{x, y_1, y_2, \cdots\})\}$ $= a$, so there exists a countable infinite compact subset $A \subseteq \bigcup_{i \in \mathbb{N}} X_i \cup \{x\}$ with a unique limit point z such that $h(T, A) = a > 0$. By assumptions, if $z = x$ then each $A_i \doteq A \cap X_i$ is a finite subset, which implies

$$h(T, A) \leq h(T, A \cup \{y_1, y_2, \cdots\})$$

$$= \max\left\{\sup_{i \in \mathbb{N}} h(T, A_i \cup \{y_i\}), h(T, \{x, y_1, y_2, \cdots\})\right\} \text{ (using Lemma 6.2)} = 0,$$

a contradiction with $h(T, A) = a > 0$. Thus $z \neq x$, and so $z \in X_v$ for some $v \in \mathbb{N}$. Put $r = \frac{d(z,x)}{2}$, then $r > 0$ follows from $x \notin \Phi_{\epsilon_v}(x_v)$. Since $\{x\}$ is the limit of $\{X_i\}_{i \in \mathbb{N}}$ (in the sense of Hausdorff metric H_d), there exists $L \in \mathbb{N}$ such that if $i > L$ then $d(z, X_i) \geq d(z, x) - H_d(X_i, \{x\}) > r$. Thus $(\bigcup_{i > L} X_i \cup \{x\}) \cap A$ is a finite set, as z is the unique limit point of A and $d(z, (\bigcup_{i > L} X_i \cup \{x\}) \cap A) \geq r$. Therefore

$$h(T, A) = h\left(T, A \cap \bigcup_{1 \leq i \leq L} X_i\right) \leq \max_{1 \leq i \leq L} h(T, X_i) = \max_{1 \leq i \leq L} a_i < a,$$

a contradiction.

Case 2. $x \in \Phi_{\epsilon_i}(x_i)$ for each large enough $i \in \mathbb{N}$. W.l.g. we assume that for each $i \in \mathbb{N}$, $x \in \Phi_{\epsilon_i}(x_i)$ and so $\Phi_{\epsilon_i}(x_i) \subseteq \Phi_{2\epsilon_i}(x)$. So $\lim_{\epsilon \to 0+} h(T, \Phi_\epsilon(x)) = a$. As $h(T, \Phi_\epsilon(T^k x)) \geq h(T, \Phi_\epsilon(x))$ for each $k \in \mathbb{N}$ and $\epsilon > 0$, then $\lim_{\epsilon \to 0+} h(T, \Phi_\epsilon(T^k x)) = a$ for each $k \in \mathbb{N}$.

If $\{x, Tx, T^2x, \cdots\}$ is infinite, we fix a point $y \in \omega(x,T) \doteq \bigcap_{n\in\mathbb{N}} \overline{\{T^j x : j \geq n\}}$. Then for each $i \in \mathbb{N}$ there exists $k_i \in \mathbb{N}$ such that $0 < d(y, T^{k_i}x) < \frac{1}{i}$. For each $i \in \mathbb{N}$, as $\lim_{\epsilon\to 0+} h(T, \Phi_\epsilon(T^{k_i}x)) = a$, we may take $0 < \eta_i < d(y, T^{k_i}x)$ such that $h(T, \Phi_{\eta_i}(T^{k_i}x)) > \min\{a(1-\frac{1}{i}), i\}$. Let $y_i = T^{k_i}x$, then $\lim_{i\to+\infty} y_i = y$ and $\lim_{i\to+\infty} h(T, \phi_{\eta_i}(y_i)) = a$ and $y \notin \Phi_{\eta_i}(y_i)$ for each $i \in \mathbb{N}$. By a similar proof to Case 1, it is impossible. Hence, x must have a finite orbit, we may assume that x is a periodic point (if necessary we replace x by $T^k x$ for some $k \in \mathbb{N}$).

Let $l \in \mathbb{N}$ be the period of x. Since $T(\Phi_\epsilon(T^k x)) \subseteq \Phi_\epsilon(T^{k+1}x)$ for each $k \in \mathbb{N}$ and $\epsilon > 0$, $\bigcup_{i=0}^{l-1} \Phi_\epsilon(T^i x)$ is compact and T-invariant (i.e. $T(\bigcup_{i=0}^{l-1} \Phi_\epsilon(T^i x)) \subseteq \bigcup_{i=0}^{l-1} \Phi_\epsilon(T^i x))$ for any $\epsilon > 0$. For each $n \in \mathbb{N}$, let $Y_n = \bigcup_{i=0}^{l-1} \Phi_{\frac{1}{n}}(T^i x)$. Then (Y_n, T) is a sub-system of (X,T) and $h_{\text{top}}(T, Y_n) \geq h(T, \Phi_{\frac{1}{n}}(x)) \geq a$. By the variational principle, there exists $\mu_n \in \mathcal{M}^e(Y_n, T)$ such that $h_{\mu_n}(T, Y_n) > \min\{a(1-\frac{1}{n}), n\}$. Obviously, $\mu_n(\{x\}) = 0$ and $\mu_n(\Phi_{\frac{1}{n}}(x)) \geq \frac{1}{l}$. Thus, we can take a compact subset $K_n \subseteq \Phi_{\frac{1}{n}}(x)$ such that $\mu_n(K_n) > 0$ and $x \notin K_n$. By a classic result of Katok [16, Theorem 1.1] (see also [30, Theorem 3.7]), we know that

$$h(T, K_n) \geq h_{\mu_n}(T, Y_n) > \min\left\{a\left(1 - \frac{1}{n}\right), n\right\},$$

as $\mu_n(K_n) > 0$. As (X,T) is **HUL**, there is a countable compact subset $X_n \subseteq K_n$ with a unique limit point y_n such that $h(T, X_n) = a_n = \min\{a(1-\frac{1}{n}), n\} < a$. Clearly, $x \notin X_n$ for each $n \in \mathbb{N}$, as $x \notin K_n$. Again by a similar proof to Case 1, we know that this is impossible. Thus, (X,T) must be asymptotically h-expansive. $\square$

7. Properties preserved by a principal extension

Combined with the results obtained in sections 5 and 6, in this section we shall answer question 1.4 by proving that a TDS is **HUL** iff it is asymptotically h-expansive. Moreover, we present a hereditarily lowerable TDS with finite entropy which is not **HUL**. As a byproduct, we show that principal extension preserves the properties of lowering, hereditary lowering and **HUL**.

Let (X,T) and (Y,S) be GTDSs. We say that $\pi : (X,T) \to (Y,S)$ is a *factor map* if π is a continuous surjective map and $\pi \circ T = S \circ \pi$. Let $\pi : (X,T) \to (Y,S)$ be a factor map between GTDSs. The *relative topological entropy of* (X,T) *w.r.t.* π is defined as follows:

$$h_{\text{top}}(T, X|\pi) = \sup_{y\in Y} h(T, \pi^{-1}(y)).$$

Let $\pi : (X,T) \to (Y,S)$ be a factor map between GTDSs. It's easy to check that on Y the function $y \mapsto h(T, \pi^{-1}(y))$ is S-invariant and Borel measurable. Thus for each $\nu \in \mathcal{M}(Y,S)$ we may define

$$h(T, X|\nu) = \int_Y h(T, \pi^{-1}(y)) d\nu(y).$$

In particular, if ν is ergodic then $h(T, \pi^{-1}(y)) = h(T, X|\nu)$ for ν-a.e $y \in Y$. Thus

Proposition 7.1. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between GTDSs. Then for each $\nu \in \mathcal{M}^e(Y, S)$ there exists a countable compact set $K \in 2^X$ with $h(T, K) = h(T, X|\nu)$.*

Now let $\pi : (X, T) \to (Y, S)$ be a factor map between GTDSs and $\mu \in \mathcal{M}(X, T)$, note that the sub-$\sigma$-algebra $\pi^{-1}(\mathcal{B}_Y) \subseteq \mathcal{B}_X$ satisfies $T^{-1}\pi^{-1}(\mathcal{B}_Y) \subseteq \pi^{-1}(\mathcal{B}_Y)$ in the sense of μ, we define *relative measure-theoretical μ-entropy of (X, T) w.r.t. π* as

$$h_\mu(T, X|\pi) = h_\mu(T, X|\pi^{-1}(\mathcal{B}_Y)).$$

The following results are proved in [10] and [19].

Lemma 7.2. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between GTDSs. Then*

(1) *One has*

$$h_{top}(T, X|\pi) = \sup_{\nu \in \mathcal{M}(Y,S)} h(T, X|\nu) = \sup_{\nu \in \mathcal{M}^e(Y,S)} h(T, X|\nu).$$

(2) *For each $\nu \in \mathcal{M}(Y, S)$,*

$$h(T, X|\nu) = \sup\{h_\mu(T, X|\pi) : \mu \in \mathcal{M}(X, T), \pi\mu = \nu\}.$$

(3) *For each $\mu \in \mathcal{M}(X, T)$, $h_\mu(T, X) = h_\mu(T, X|\pi) + h_{\pi\mu}(S, Y)$.*

We have proved that each expansive TDS is **HUL**. In fact, the same conclusion holds for a more general case. To prove this, first we shall prove the following Bowen's type theorem which is interesting itself. We remark that the idea of the proof is inspired by the proof of [2, Theorem 17].

Theorem 7.3. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between GTDSs and $E \in 2^X$. Then*

$$h(S, \pi(E)) \leq h(T, E) \leq h(S, \pi(E)) + h_{top}(T, X|\pi).$$

In particular, if (Y, S) has finite entropy then for each $E \in 2^X$,

$$h(T, E) - h(S, \pi(E)) \leq h_{top}(T, X|\pi).$$

Proof. By the continuity of π, it's easy to obtain $h(S, \pi(E)) \leq h(T, E)$. Thus it remains to prove $h(T, E) \leq h(S, \pi(E)) + h_{\mathrm{top}}(T, X|\pi)$. If $h_{\mathrm{top}}(T, X|\pi) = +\infty$, this is obvious. Now we suppose that $a = h_{\mathrm{top}}(T, X|\pi) < +\infty$. Let d_X and d_Y be the metrics on (X, T) and (Y, S), respectively.

Let $\epsilon > 0$ and $\alpha > 0$. By Lemma 7.2 (1), for each $y \in Y$ we may choose $m(y) \in \mathbb{N}$ such that

$$a + \alpha \geq h(T, \pi^{-1}(y)) + \alpha \geq \frac{1}{m(y)} \log r_{m(y)}(d_X, T, \epsilon, \pi^{-1}(y)). \tag{7.1}$$

Let E_y be an $(m(y), \epsilon)$-spanning set of $\pi^{-1}(y)$ w.r.t. T with the minimum cardinality. Then $U_y \doteq \bigcup_{z \in E_y} B_{m(y)}(z, 2\epsilon)$ is an open neighborhood of $\pi^{-1}(y)$, where $B_{m(y)}(z, 2\epsilon)$ denotes the open ball in X with center z and radius 2ϵ (in the sense of $(d_X)_{m(y)}$-metric). Since the map $\pi^{-1} : Y \to 2^X$, $y \mapsto \pi^{-1}(y)$ is upper semi-continuous, there exists an open neighborhood W_y of y for which $\pi^{-1}(W_y) \subseteq U_y$. By the compactness

of Y there exist $\{y_1, \cdots, y_k\} \subseteq Y$ such that $\mathcal{W} \doteq \{W_{y_1}, \cdots, W_{y_k}\}$ forms an open cover of Y. Let $\delta > 0$ be a Lebesgue number of $\mathcal{W}$ and $M = \max\{m(y_1), \cdots, m(y_k)\}$.

Let $\pi(E)_n$ be any (n, δ)-spanning set for $\pi(E)$ w.r.t. S with the minimum cardinality. For each $y \in \pi(E)_n$ and $0 \leq j < n$, pick $c_j(y) \in \{y_1, \cdots, y_k\}$ with $\overline{B(S^j(y), \delta)} \subseteq W_{c_j(y)}$, where $B(S^j(y), \delta)$ denotes the open ball in Y with center $S^j(y)$ and radius δ. Now define recursively $t_0(y) = 0$ and $t_{s+1}(y) = t_s(y) + m(c_{t_s(y)}(y))$ $(s \in \mathbb{Z}_+)$ until one gets a $t_{q+1}(y) \geq n$; set $q(y) = q \leq t_q(y)$.

For any $y \in \pi(E)_n$ and $z_0 \in E_{c_{t_0(y)}(y)}, z_1 \in E_{c_{t_1(y)}(y)}, \cdots, z_{q(y)} \in E_{c_{t_{q(y)}(y)}(y)}$ we define

$$V(y; z_0, z_1, \cdots, z_{q(y)}) = \{u \in X : (d_X)_{m(c_{t_s(y)}(y))}(T^{t_s(y)}(u), z_s) < 2\epsilon, 0 \leq s \leq q(y)\}.$$

Obviously, for each $y \in \pi(E)_n$, the number of permissible tuples $(z_0, z_1, \cdots, z_{q(y)})$ is

$$N_y = \prod_{s=0}^{q(y)} r_{m(c_{t_s(y)}(y))}(d_X, T, \epsilon, \pi^{-1}(c_{t_s(y)}(y))). \tag{7.2}$$

Then we have (using (7.1) and (7.2))

$$N_y \leq \prod_{s=0}^{q(y)} e^{(a+\alpha)m(c_{t_s(y)}(y))} = e^{(a+\alpha)(t_{q(y)}(y)+m(c_{t_{q(y)}(y)}(y)))} \leq e^{(a+\alpha)(n+M)}. \tag{7.3}$$

Note that if F is an $(n, 4\epsilon)$-separated subset of E w.r.t. T then, for each permissible tuple $(z_0, z_1, \cdots, z_{q(y)})$, $V(y; z_0, z_1, \cdots, z_{q(y)}) \cap F$ has at most one element, and

$$\bigcup_{y \in \pi(E)_n} \left(\bigcup_{z_s \in E_{c_{t_i(y)}(y)}, 0 \leq s \leq q(y)} V(y; z_0, z_1, \cdots, z_{q(y)}) \right) \supseteq E.$$

Thus combining (7.3) we have

$$s_n(d_X, T, 4\epsilon, E) \leq \sum_{y \in \pi(E)_n} N_y \leq r_n(d_Y, S, \delta, \pi(E)) e^{(a+\alpha)(n+M)}.$$

Letting $n \to +\infty$ one has $s(d_X, T, 4\epsilon, E) \leq r(d_Y, S, \delta, \pi(E)) + a + \alpha \leq h(S, \pi(E)) + a + \alpha$. Since $\epsilon > 0$ and $\alpha > 0$ are arbitrary, we obtain $h(T, E) \leq h(S, \pi(E)) + a$. This finishes the proof. $\square$

As a direct consequence of Theorem 7.3, we have the following proposition.

Proposition 7.4. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between GTDSs. If (Y, S) has finite entropy, then*

$$\sup_{E \in 2^X} (h(T, E) - h(S, \pi(E))) = h_{top}(T, X|\pi).$$

Proof. On one hand we know $\sup_{E\in 2^X}(h(T,E)-h(S,\pi(E))\le h_{\text{top}}(T,X|\pi)$ by Theorem 7.3. On the other hand, we have

$$\begin{aligned}\sup_{E\in 2^X}(h(T,E)-h(S,\pi(E))&\ge \sup_{y\in Y}(h(T,\pi^{-1}(y))-h(S,\{y\}))\\&=\sup_{y\in Y}h(T,\pi^{-1}(y))=h_{\text{top}}(T,X|\pi).\end{aligned}$$

This completes the proof. □

Let $\pi:(X,T)\to(Y,S)$ be a factor map between GTDSs. We call that π is a *principal factor map* (or (X,T) is a *principal extension of* (Y,S)) if $h_\mu(T,X)=h_{\pi\mu}(S,Y)$ for each $\mu\in\mathcal{M}(X,T)$. This was introduced and studied firstly by Ledrappier [18].

The following result is a direct consequence of Lemma 7.2 and Theorem 7.3.

Corollary 7.5. *Let $\pi:(X,T)\to(Y,S)$ be a principal factor map betweens GTDSs. If (Y,S) has finite entropy then $h_{top}(T,X|\pi)=0$ and $h(T,K)=h(S,\pi(K))$ for all $K\in 2^X$.*

A characterization of asymptotical h-expansiveness is obtained recently by Boyle and Downarowicz as the following [5, Theorem 8.6]:

Lemma 7.6. *A TDS (X,T) is asymptotically h-expansive iff it admits a principal extension to a symbolic TDS.*

Then question 1.4 is answered as follows:

Theorem 7.7. *A TDS (X,T) is asymptotically h-expansive iff it is* **HUL**.

Proof. First, each **HUL** TDS is asymptotically h-expansive by Theorem 6.4.

Now assume that TDS (X,T) is asymptotically h-expansive and by Lemma 7.6 let $\pi:(X',T')\to(X,T)$ be a principal factor map with (X',T') a symbolic TDS. Then $h_{\text{top}}(T,X)\le h_{\text{top}}(T',X')<+\infty$ (as (X',T') is a symbolic TDS), and so for each $E\in 2^{X'}$, we have $h(T',E)=h(T,\pi(E))$ by Corollary 7.5. Given $E\in 2^X$. Since (X',T') is an expansive TDS, then using Theorem 5.4 we have that for each $0\le h\le h(T,E)=h(T',\pi^{-1}(E))$ there exists a countable compact subset $X'_h\subseteq\pi^{-1}(E)$ with at most a limit point in X' such that $h(T',X'_h)=h$. Now set $X_h=\pi(X'_h)\subseteq E$. So $X_h\subseteq X$ is a countable compact subset with at most a limit point in X and $h(T,X_h)=h(T',X'_h)=h$. That is, TDS (X,T) is **HUL**. □

Moreover, combining with Corollary 7.5 and Theorem 7.7 we claim that principal extension preserves the properties of lowering, hereditary lowering and **HUL**.

Proposition 7.8. *Let $\pi:(X',T')\to(X,T)$ be a principal factor map between TDSs. If (X,T) has finite entropy then*

1. *(X',T') is asymptotically h-expansive iff so is (X,T).*
2. *(X',T') is lowerable (resp. hereditarily lowerable,* **HUL***) iff so is (X,T).*

Proof. (1) is only a special case of Ledrappier's result about principal extensions [18, Thorem 3]. (2) follows directly from Corollary 7.5, Theorem 7.7 and (1). □

It is not hard to construct examples with infinite entropy which are hereditarily lowerable. Thus, there are TDSs which are hereditarily lowerable but not **HUL**. In fact, an example with the same property which has finite entropy exists.

Example 7.9. *There exists a hereditarily lowerable TDS (X,T) with finite entropy which is not* **HUL**. *The detailed construction is given as follows:*

Take a countable copies of the full shift over $\{0,1\}^{\mathbb{Z}}$ and embed them into B_n with $\{B_n\}_{n\in\mathbb{N}}$ a sequence of disjoint compact balls in $\mathbb{R}^2$ such that $(0,0)\notin B_n\to\{(0,0)\}$ (in the sense of Hausdorff metric). Let (X,T) be the TDS of the union of $\{(0,0)\}$ with these copies, where T is the shift if it is restricted on each copy and $(0,0)\mapsto(0,0)$. Then $h_{\text{top}}(T,X)=\log 2$ (using the classical variational principle). For each copy we may take $C_n\subseteq B_n$ with entropy $a_n<\log 2$ such that $\lim_{n\to+\infty}a_n=\log 2$ (using Theorem 4.4). Then $h(T,\bigcup_{n\in\mathbb{N}}C_n\cup\{(0,0)\})=\log 2$. Whereas, by definition it is not hard to see that any countable compact subset of X with a unique limit point $(0,0)$ must have zero entropy, which implies that each countable compact subset of $\bigcup_{n\in\mathbb{N}}C_n\cup\{(0,0)\}$ with a unique limit point has entropy smaller strictly than $\log 2$. Thus (X,T) is not **HUL**.

Now we claim that (X,T) is hereditarily lowerable. For each $n\in\mathbb{N}$, we take $x_n\in B_n$. Then it is not hard to see that $h(T,\{x_n\}_{n\in\mathbb{N}}\cup\{(0,0)\})=0$. Let $K\in 2^X$ and $0\le h\le h(T,K)$. For each $n\in\mathbb{N}$, set $K_n=K\cap B_n$. Since B_n is **HUL** (see Theorem 5.4), we may take $K_n^h\in 2^{K_n}$ with $h(T,K_n^h)=\min\{h,h(T,K_n)\}$. Using Lemma 6.2 we have

$$\begin{aligned}\sup_{n\in\mathbb{N}}h(T,K_n) &= \max\left\{\sup_{n\in\mathbb{N}}h(T,K_n\cup\{x_n\}),h(T,\{x_n\}_{n\in\mathbb{N}}\cup\{(0,0)\})\right\}\\ &= h\left(T,\bigcup_{n\in\mathbb{N}}(K_n\cup\{x_n\})\cup\{(0,0)\}\right)\ge h(T,K)\ge h.\end{aligned}$$

This implies $\sup_{n\in\mathbb{N}}h(T,K_n^h)=h$. Let $K_h=\bigcup_{n\in\mathbb{N}}K_n^h\cup\{(0,0)\}\in 2^K$. Then using Lemma 6.2 again one has $h(T,K_h)=\sup_{n\in\mathbb{N}}h(T,K_n^h)=h$. This means that K is lowerable, and so (X,T) is a hereditarily lowerable TDS. This ends the example.

It is not difficult to show that the above example (by a small modification) has a symbolic extension with the same entropy, which is not a principal one (see [5] for other examples of the same type). Thus it is an interesting question if each system having a symbolic extension is hereditarily lowerable.

8. Appendix

In this Appendix we want to explain that our main results hold for GTDSs. Note that we can also introduce the lowerable, hereditarily lowerable, **HUL** and asymptotically h-expansive properties for a GTDS. Let (X,T) be a GTDS. If T is surjective, we can use the standard natural extension as follows:

Assume that d is a metric on X. We say (X_T,S) is the *natural extension* of (X,T), if $X_T=\{(x_1,x_2,\cdots):T(x_{i+1})=x_i,x_i\in X,i\in\mathbb{N}\}$, which is a sub-space of the

product space $\prod_{i=1}^{\infty} X$ with the compatible metric d_T defined by

$$d_T((x_1, x_2, \cdots), (y_1, y_2, \cdots)) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^i}.$$

Moreover, $S : X_T \to X_T$ is the shift homeomorphism, i.e. $S(x_1, x_2, x_3, \cdots) = (T(x_1), x_1, x_2, \cdots)$. The following observation is easy.

Theorem 8.1. *Let (X_T, S) be the natural extension of (X, T) with T surjective. Then (X_T, S) is lowerable (resp. hereditarily lowerable,* **HUL***, asymptotically h-expansive) iff so is (X, T).*

Proof. Let $\pi_1 : X_T \to X$ be the projection to the first coordinate. Observe that $\mathrm{diam}(S^n \pi_1^{-1}(x)) \to 0$ for each $x \in X$. This implies that $h(S, \pi_1^{-1}(x)) = 0$ for each $x \in X$, and hence $h_{\mathrm{top}}(S, X_T|\pi_1) = \sup_{x \in X} h(S, \pi_1^{-1}(x)) = 0$. By Theorem 7.3, (X_T, S) is lowerable (resp. hereditarily lowerable, **HUL**) iff so is (X, T).

Since $h(S, \pi_1^{-1}(x)) = 0$ for each $x \in X$, π_1 is a principal extension by Lemma 7.2. Now as a principal extension preserves the property of asymptotical h-expansiveness (see [18, Theorem 3]), (X_T, S) is asymptotically h-expansive iff so is (X, T). □

If T is not surjective, we will construct a surjective system (X', T') such that the dynamical properties of (X, T) and (X', T') are 'very close' as follows:

Let $X' = X \times \{0\} \cup X \times \{\frac{1}{n} : n \in \mathbb{N}\}$. Moreover, put $T'(x, 0) = (x, 0), T'(x, \frac{1}{n+1}) = (x, \frac{1}{n})$ and $T'(x, 1) = (Tx, 1)$ for $n \in \mathbb{N}$ and $x \in X$.

It is not hard to check that (X', T') is lowerable (resp. hereditarily lowerable, **HUL**, asymptotically h-expansive) iff so is (X, T). Collecting terms, one has

Theorem 8.2. *Let (X, T) be a GTDS. Then (X, T) is* **HUL** *iff it is asymptotically h-expansive, and if (X, T) has finite entropy then it is lowerable.*

References

[1] T. Bogenschütz, *Entropy pressure, and a variational principle for random dynamical systems*, Random Comput. Dynam. **1** (1992/93), 99-116.

[2] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. **153** (1971), 401-414.

[3] R. Bowen, *Entropy-expansive maps*, Trans. Amer. Math. Soc. **164** (1972), 323-331.

[4] R. Bowen, *Topological entropy for noncompact sets*, Trans. Amer. Math. Soc. **184** (1973), 125-136.

[5] M. Boyle and T. Downarowicz, *The entropy theory of symbolic extensions*, Invent. Math. **156** (2004), 119-161.

[6] M. Boyle, D. Fiebig and U. Fiebig, *Residual entropy, conditional entropy and subshift covers*, Forum Math. **14** (2002), 713-757.

[7] J. Buzzi, *Intrinsic ergodicity of smooth interval maps*, Israel J. Math. **100** (1997), 125-161.

[8] T. Downarowicz, *Entropy of a symbolic extension of a dynamical system*, Ergod. Th. & Dynam. Sys. **21** (2001), 1051-1070.

[9] T. Downarowicz, *Entropy structure*, J. Anal. Math. **96** (2005), 57-116.

[10] T. Downarowicz and J. Serafin, *Fiber entropy and conditional variational principles in compact non-metrizable spaces*, Fund. Math. **172** (2002), 217-247.

[11] K. J. Falconer, *Fractal Geometry — Mathematical Foundations and Applications*, Chichester: John Wiley & Sons, 1990.

[12] D. Feng, *Private communication.*

[13] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton Univ. Press, 1981.

[14] E. Glasner, *Ergodic Theory via Joinings*, Mathematical Surveys and Monographs **101**, American Mathematical Society, 2003.

[15] W. Huang, X. Ye and G. H. Zhang, *A local variational principle for conditional entropy*, Ergod. Th. & Dynam. Sys. **26** (2006), 219-245.

[16] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Inst. Hautes Etudes. Sci. Publ. Math. **51** (1980), 137-173.

[17] Y. Kifer, *Ergodic Theory of Random Transformations*, Progress in Probability and Statistics **10**, Birkhäuser Boston, Inc., Boston, 1986.

[18] F. Ledrappier, *A variational principle for the topological conditional entropy*, Lecture Notes in Math. **729** (1979), Springer-Verlag, 78-88.

[19] F. Ledrappier and P. Walters, *A relativised variational principle for continuous transformations*, J. London Math. Soc. (2) **16** (1977), 568-576.

[20] E. Lindenstrauss, *Lowering topological entropy*, J. Anal. Math. **67** (1995), 231-267.

[21] E. Lindenstrauss, *Mean dimension, small entropy factors and an embedding theorem*, Inst. Hautes Etudes. Sci. Publ. Math. **89** (1999), 227-262.

[22] E. Lindenstrauss and B. Weiss, *Mean topological dimension*, Israel J. Math. **115** (2000), 1-24.

[23] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces, Fractals and Rectifiability*, Cambridge Studies in Advanced Mathematics **44**, Cambridge University Press, Cambridge, 1995.

[24] M. Misiurewicz, *Diffeomorphism without any measure with maximal entropy*, Bull. Acad. Pol. Sci. **21** (1973), 903-910.

[25] M. Misiurewicz, *Topological conditional entropy*, Studia Math. **55** (1976), 175-200.

[26] Y. Pesin, *Dimension Theory in Dynamical Systems. Contemporary Views and Applications*, Chicago Lectures in Mathematics, The University of Chicago Press, Chicago, 1998.

[27] D. Richeson and J. Wiseman, *Positively expansive dynamical systems*, Topology Appl. **154** (2007), 604-613.

[28] M. Shub and B. Weiss, *Can one always lower topological entropy*? Ergod. Th. & Dynam. Sys. **11** (1991), 535-546.

[29] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics **79**, Springer-Verlag, New York-Berlin, 1982.

[30] X. Ye and G. H. Zhang, *Entropy points and applications*, Trans. Amer. Math. Soc. **359** (2007), 6167-6186.

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